Bandelets: the Next Generation

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Geometrical Image Representation

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- Need to take advantage of geometrical image regularity to improve representations.
- Building harmonic analysis representations adapted to complex $geometry$.

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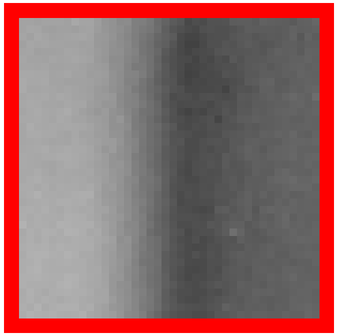

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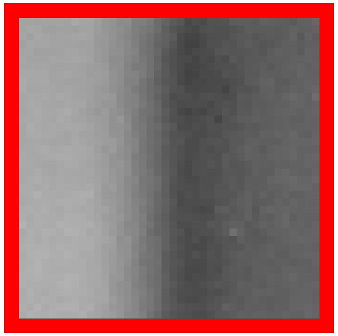

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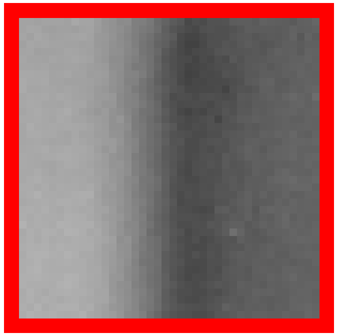

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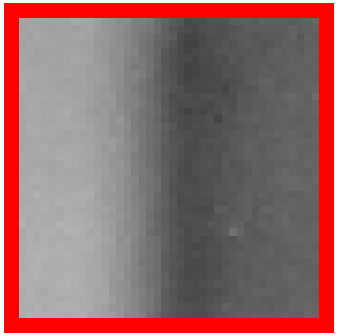

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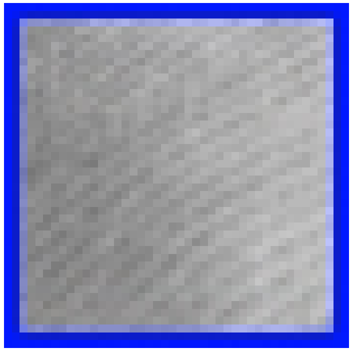

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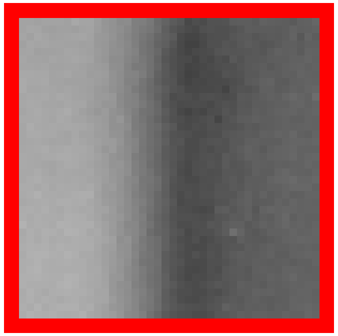 

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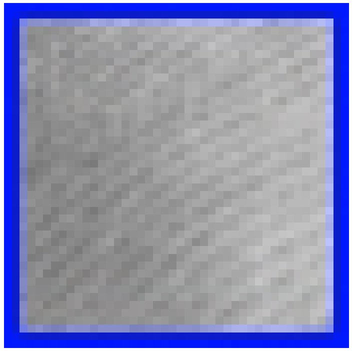 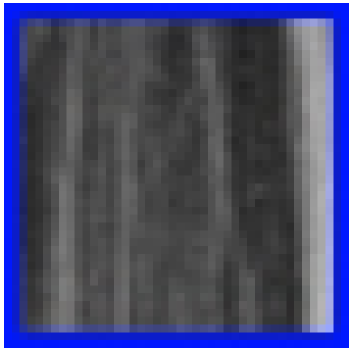

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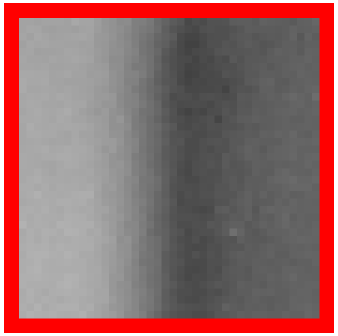


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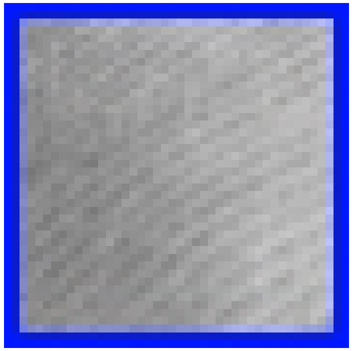
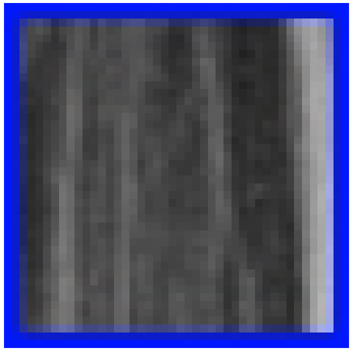
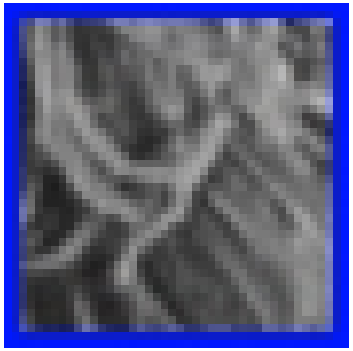

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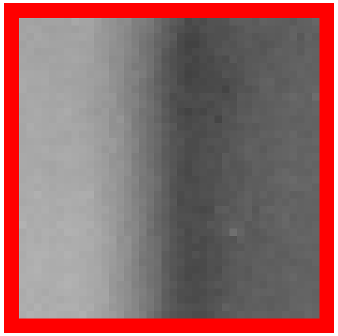

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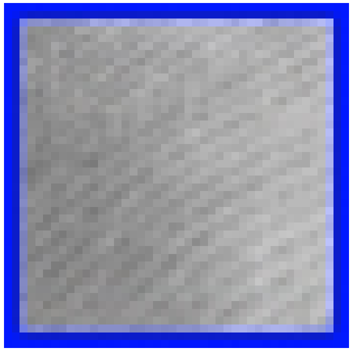
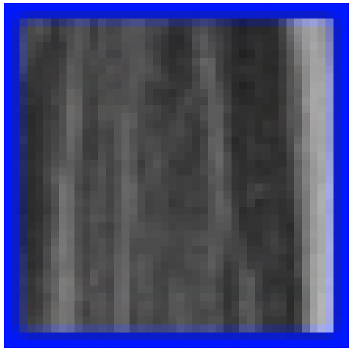
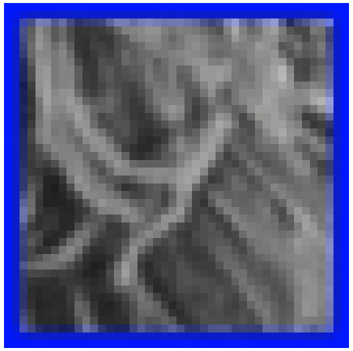
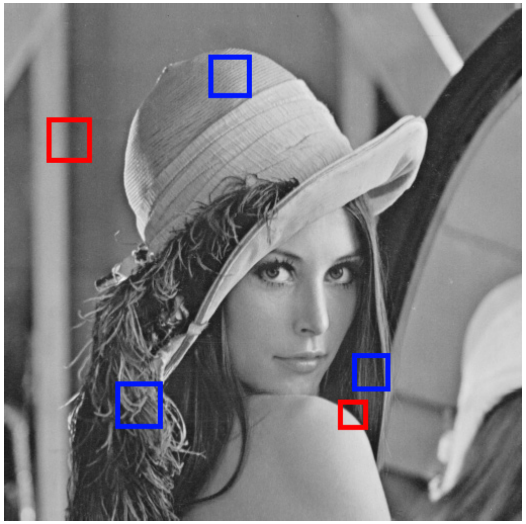






- How can the estimation of the geometry become well-posed ?



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- **Problem:** Given that $f \in \Theta$, how to choose $\mathbf{B}$ so that
$$\|f - f_M\|^2 \leqslant C M^{-\beta} \ \text{ with } \beta \text{ large ?}$$

Successes and Failures of Wavelet Bases

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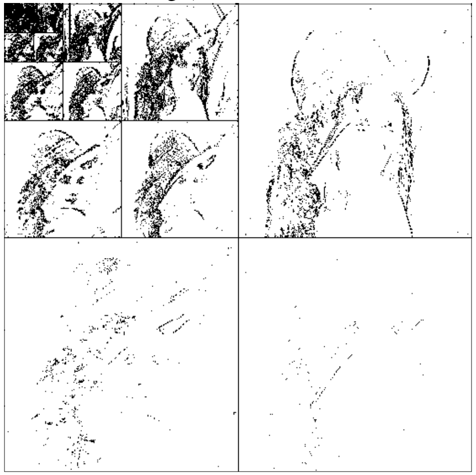

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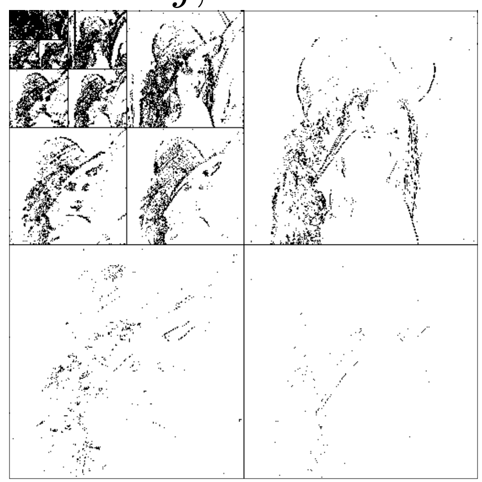

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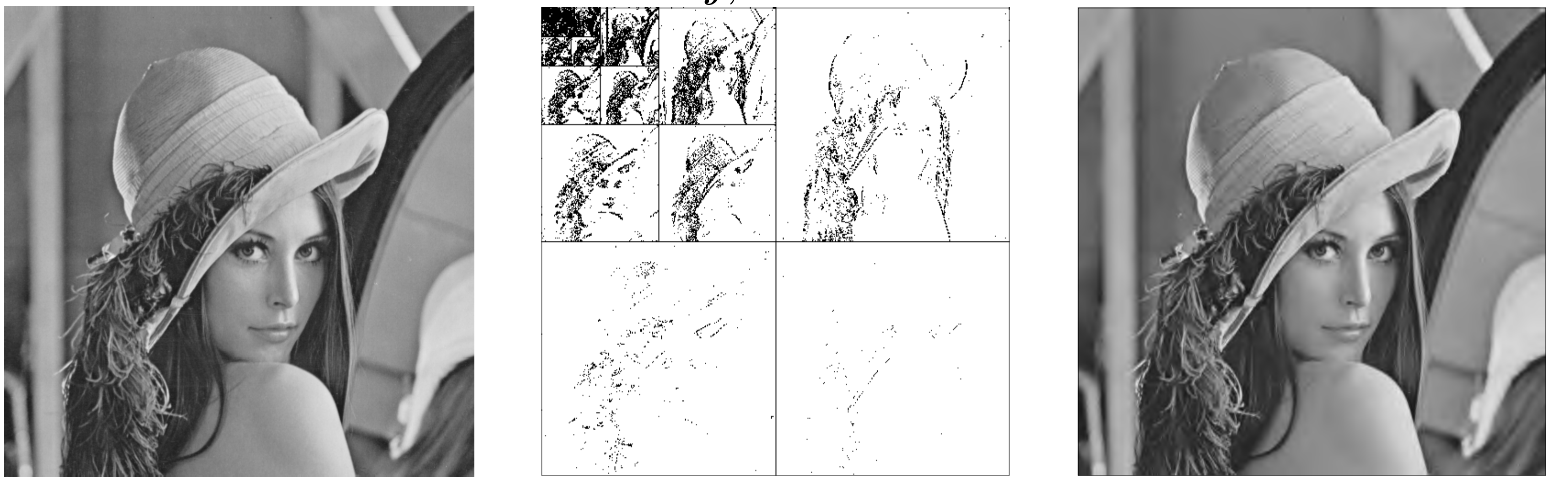

- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.
- But: does not take advantage of any geometric regularity.

Wavelet Approximation with Edges

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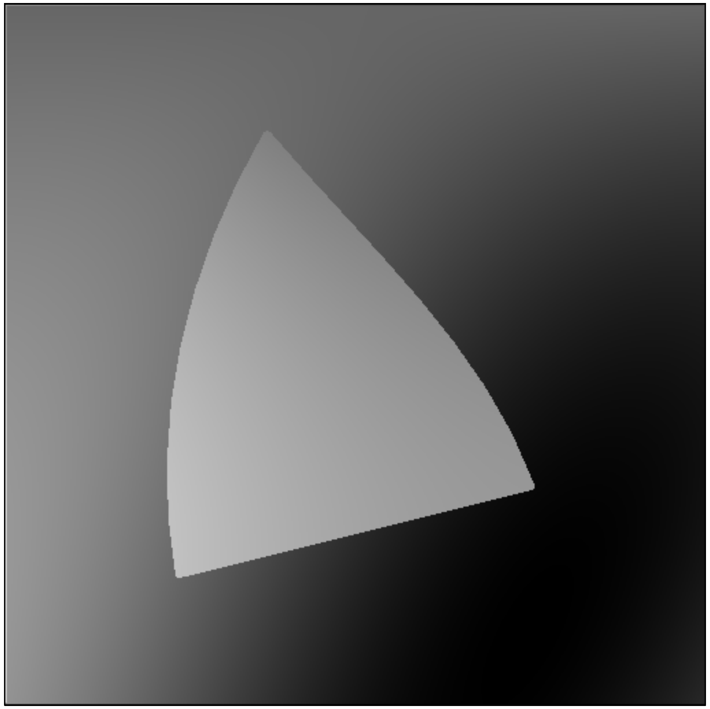
- Approximations of f which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ “edge” curves:



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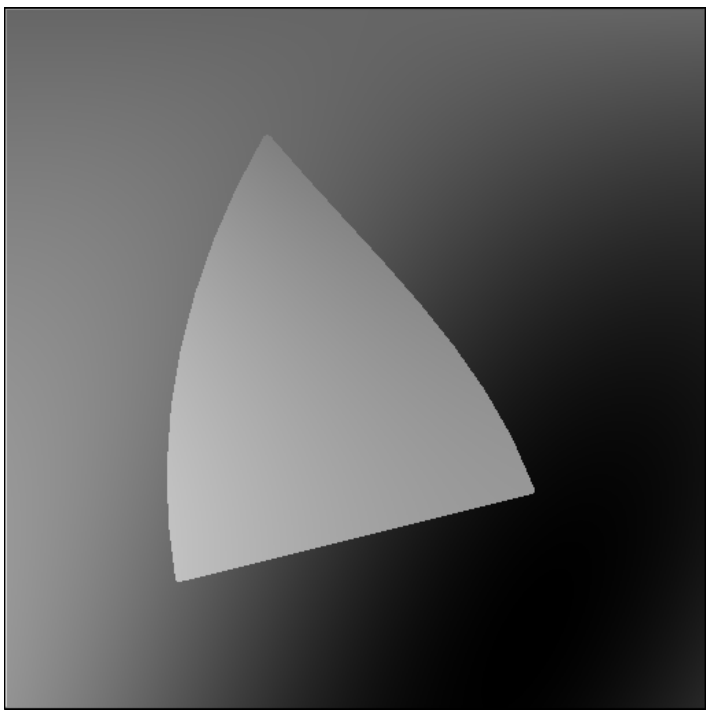
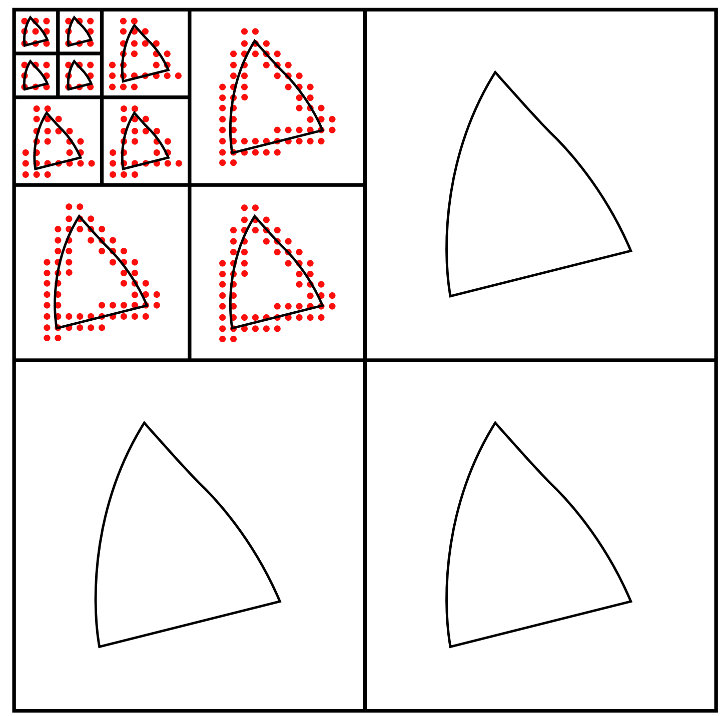


$2^j \sim T \sim M^{-1}$



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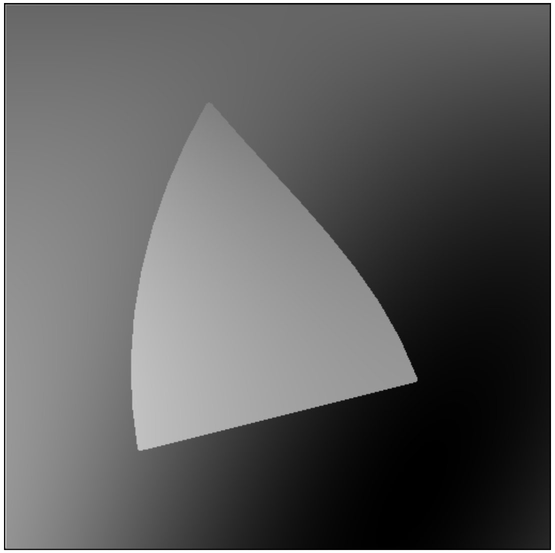

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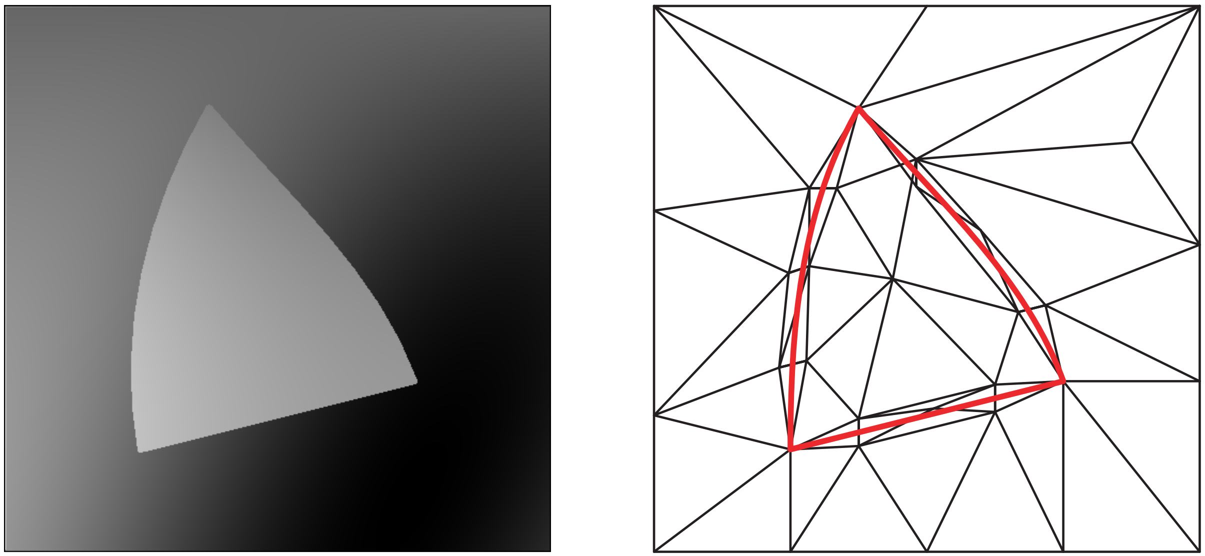
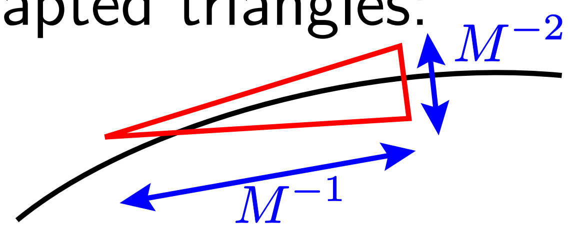


- Piecewise linear approximation over M adapted triangles: if $\alpha \geqslant 2$ then $\|f - f_M\|^2 \leqslant C\, M^{-2}$,



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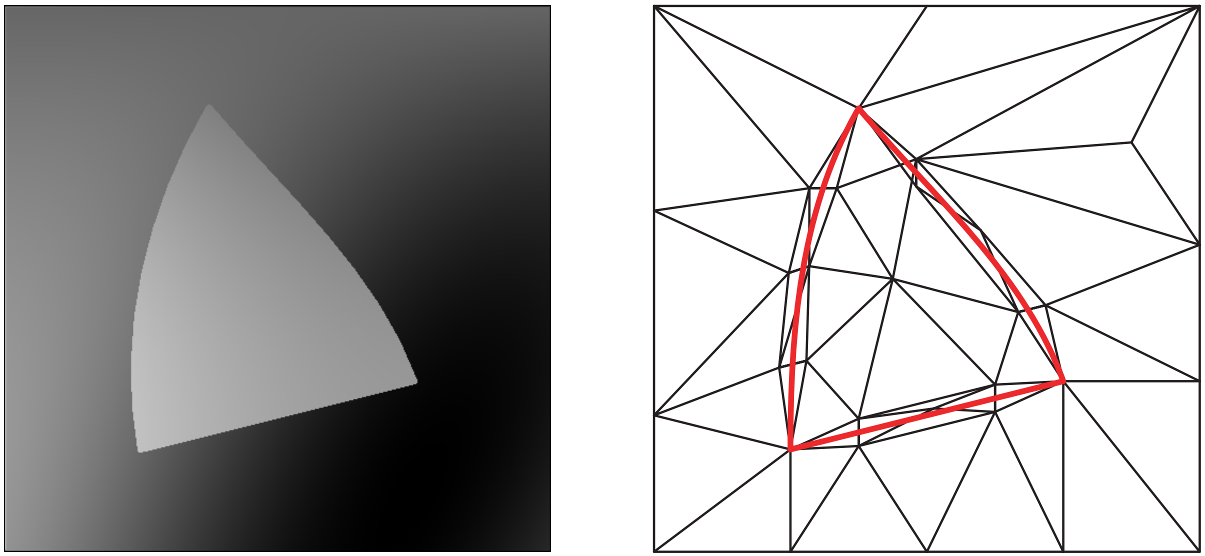

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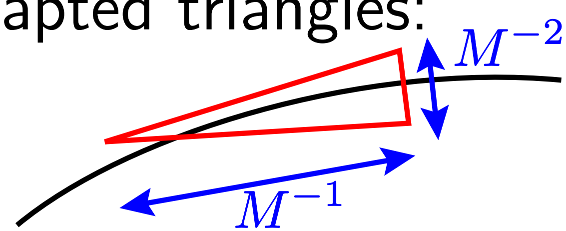
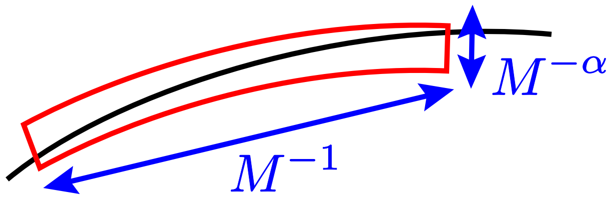


- Higher order approximation over M adapted "elements": $\|f - f_M\|^2 \leqslant C\,M^{-\alpha}$.



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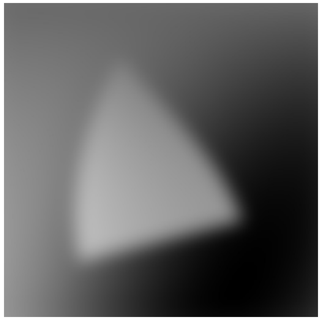

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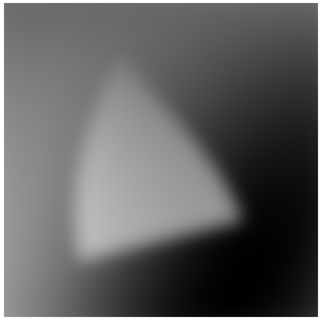

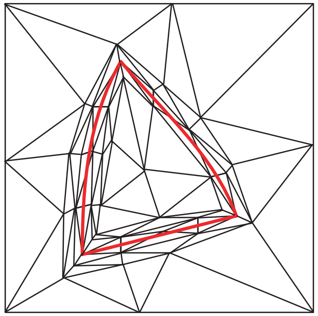

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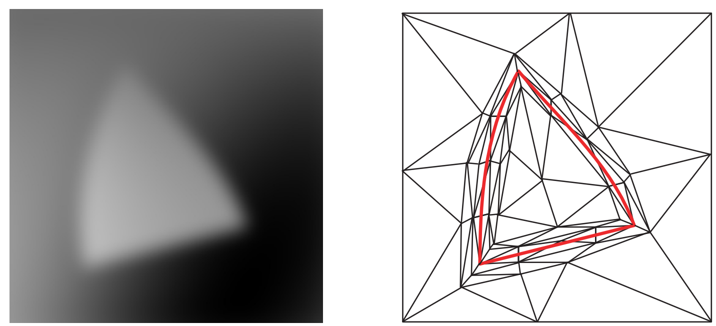

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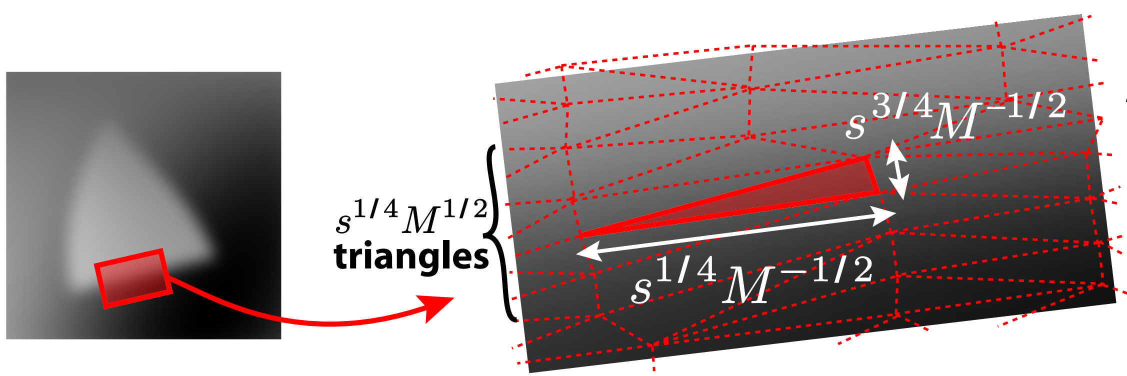

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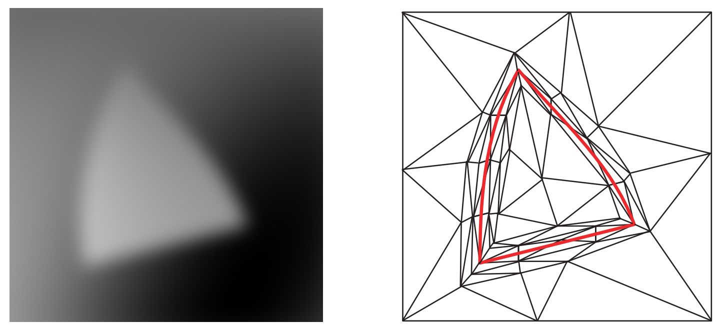

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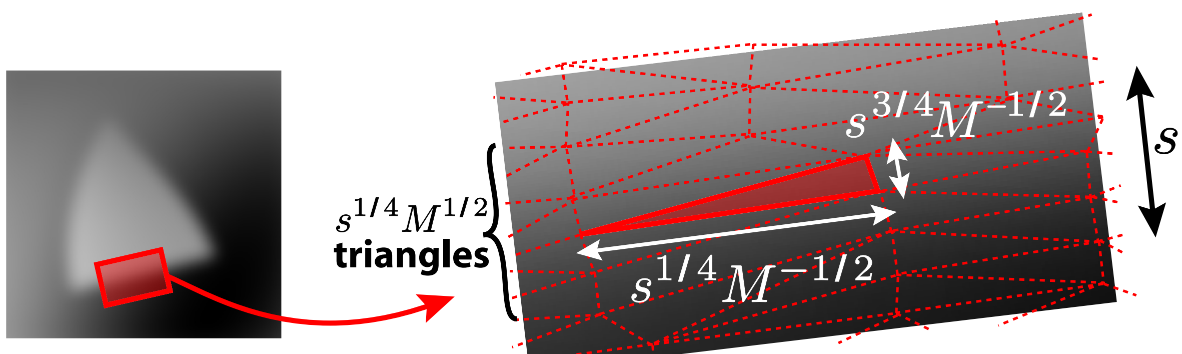

- Difficult to find optimal approximations but good greedy solutions (*Dekel,Demaret, Dyn, Iske*).

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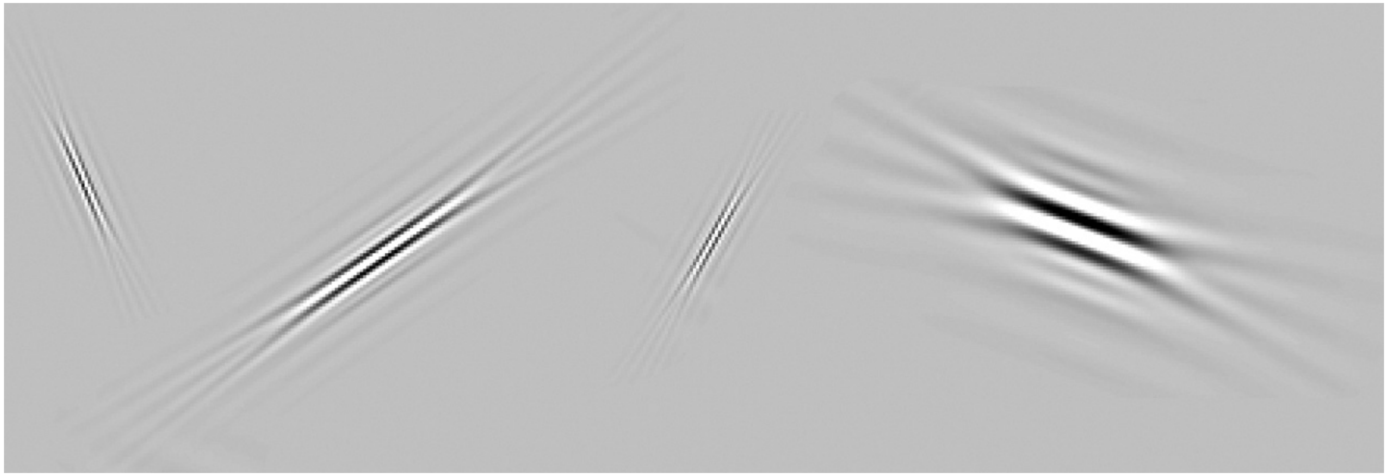

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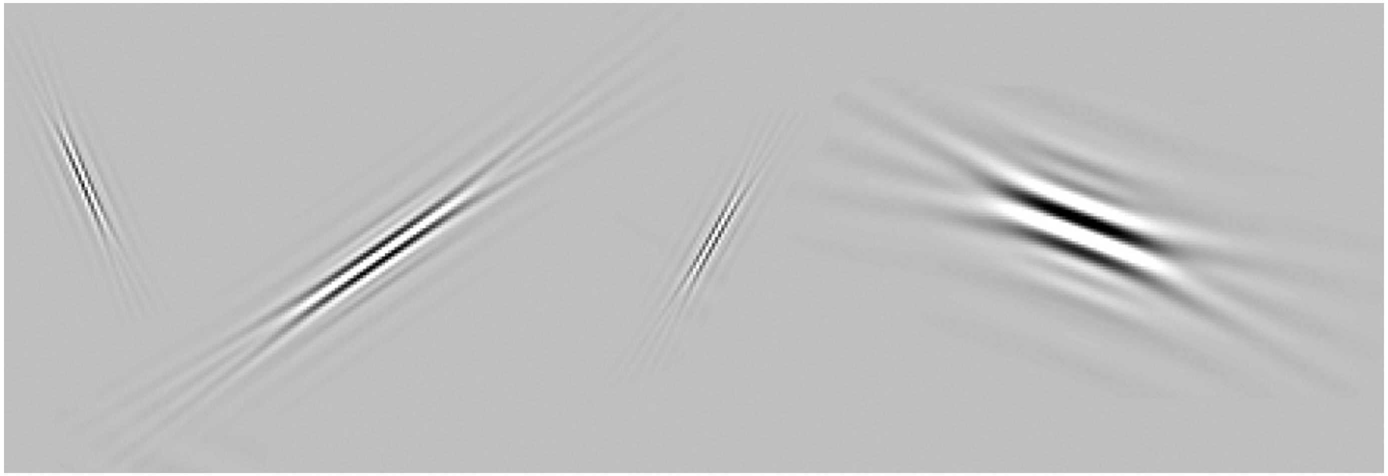

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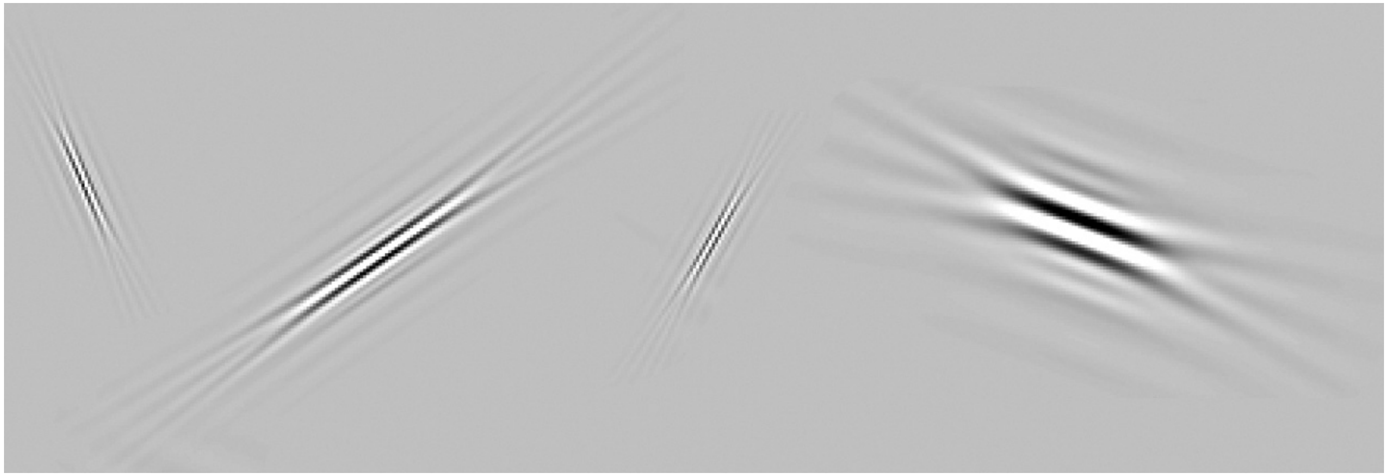

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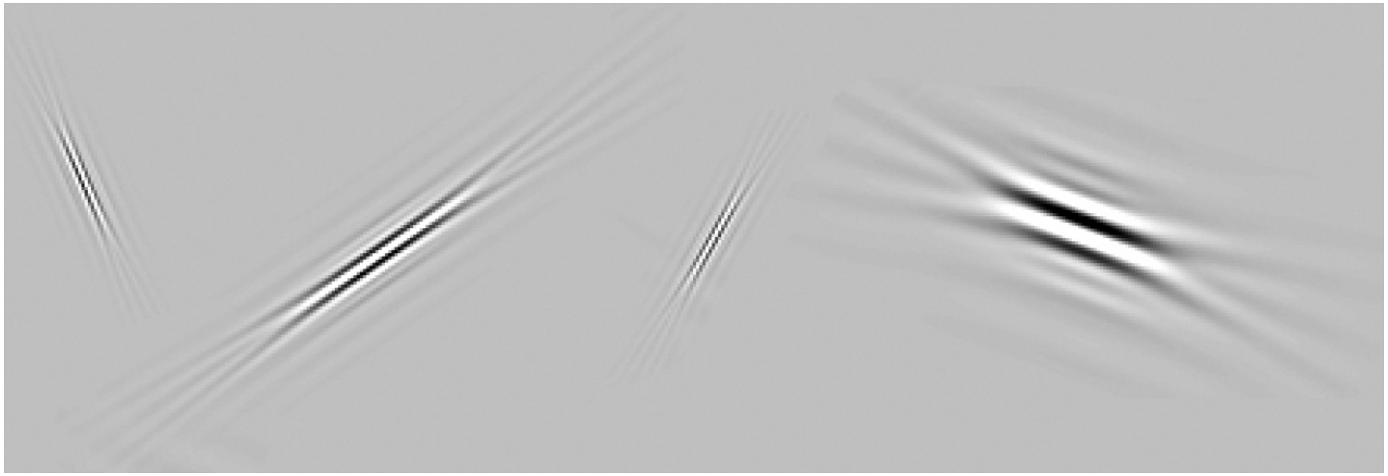

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- Optimal for $\alpha = 2$.
- Difficulty to build discrete orthogonal/Riesz bases: (*Vetterli & Minh Do*).

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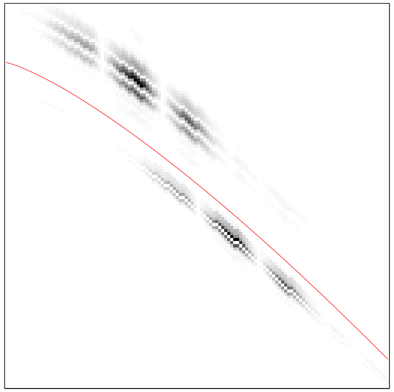

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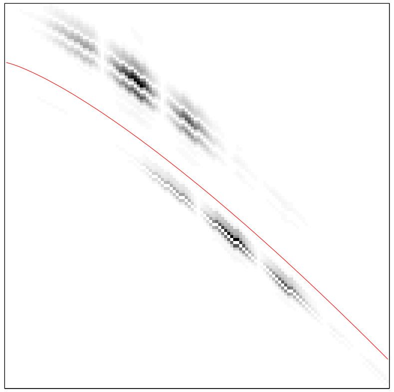

- Basis adapted to the geometry: bandelets with an anisotropic support that follows the direction of regularity of the image,

$$\left\{ \frac{1}{2^{(j+l)/2}} \Psi^d \left(\frac{x_1 - 2^l m_1}{2^l}, \frac{x_2 - c(x_1) - 2^j m_2}{2^j} \right) \right\}_{d,j,l \geqslant j, m_1, m_2}.$$

- Dyadic segmentation of the geometry: bandelet basis adapted to an image.

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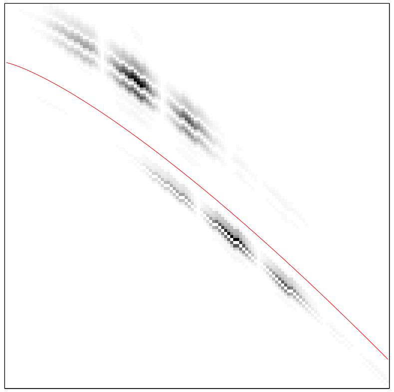 

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- Efficient optimization of this geometry: non linear approximation theorem.

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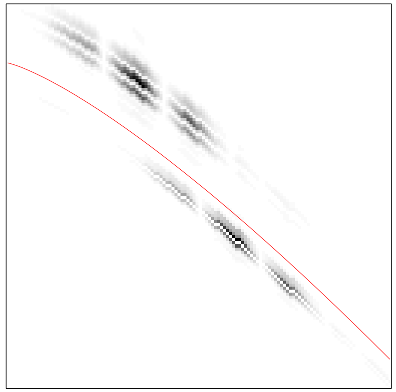 

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- Lack of a multiscale geometry.

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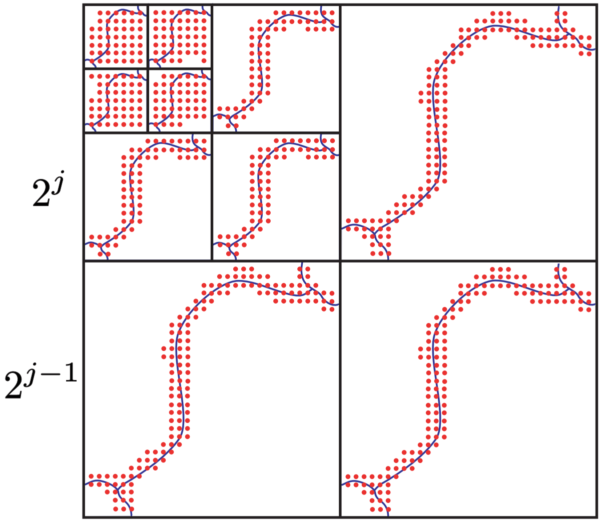

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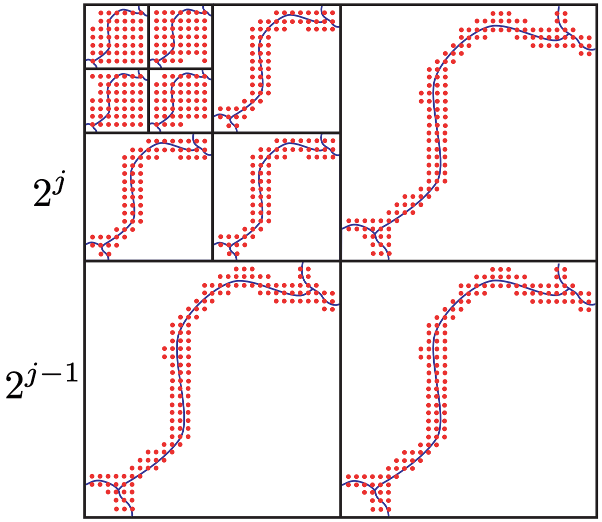
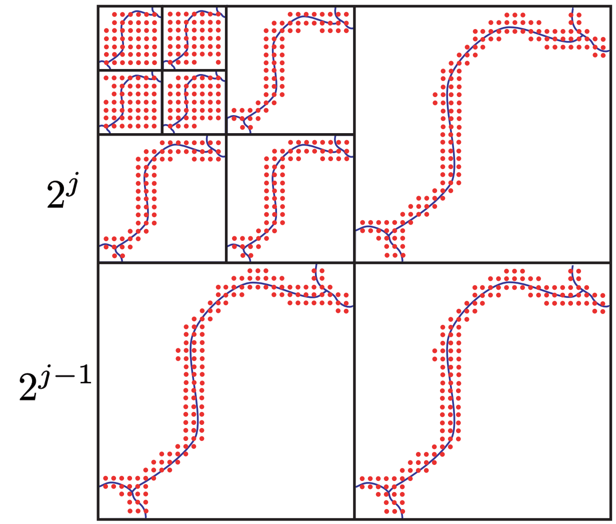


- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?

Return to Wavelet Coefficients



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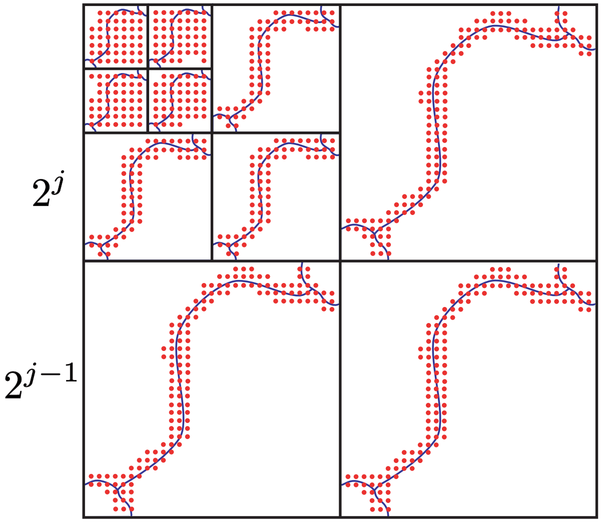

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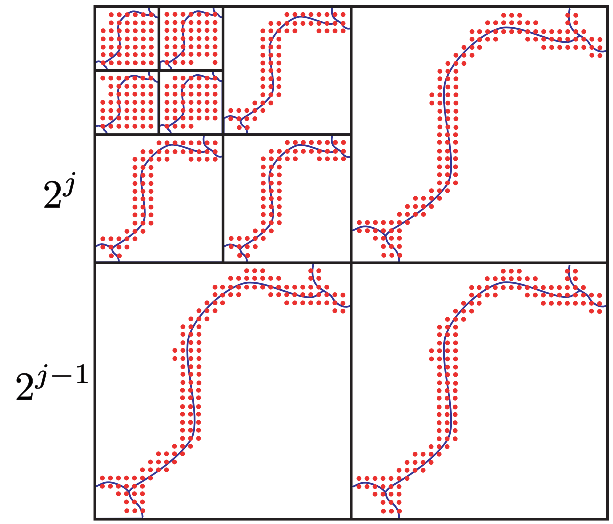

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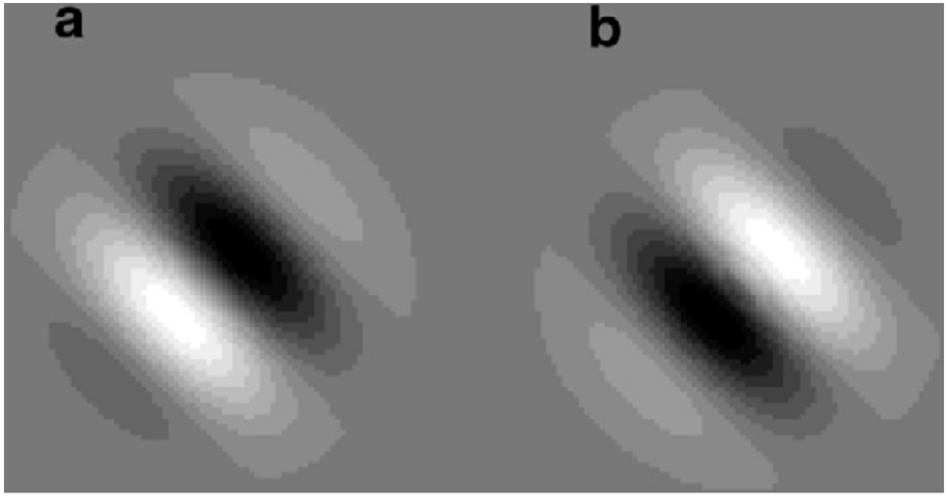
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.
- Modification of the wavelet transform (*Cohen*).

Geometry in the Visual Brain

Geometry in the Visual Brain

- Simple cells in V1 provide inner products with wavelets:



[*Wolf et al*]

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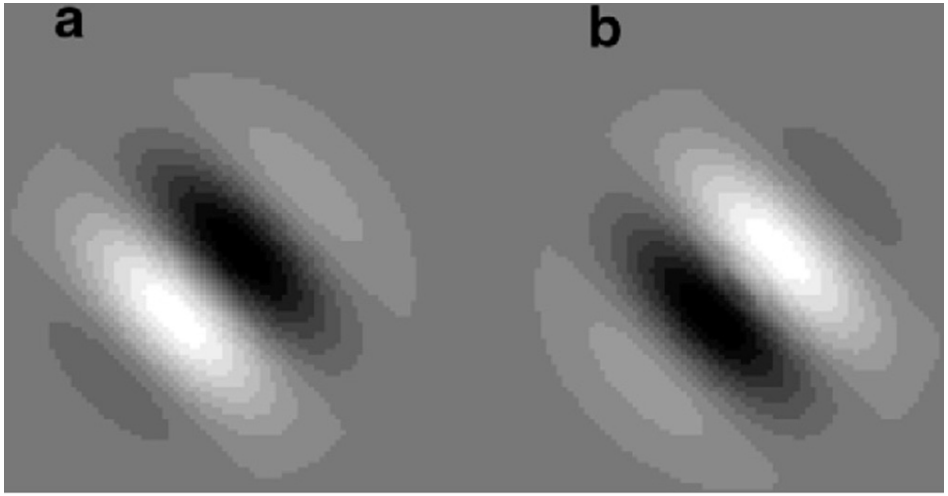

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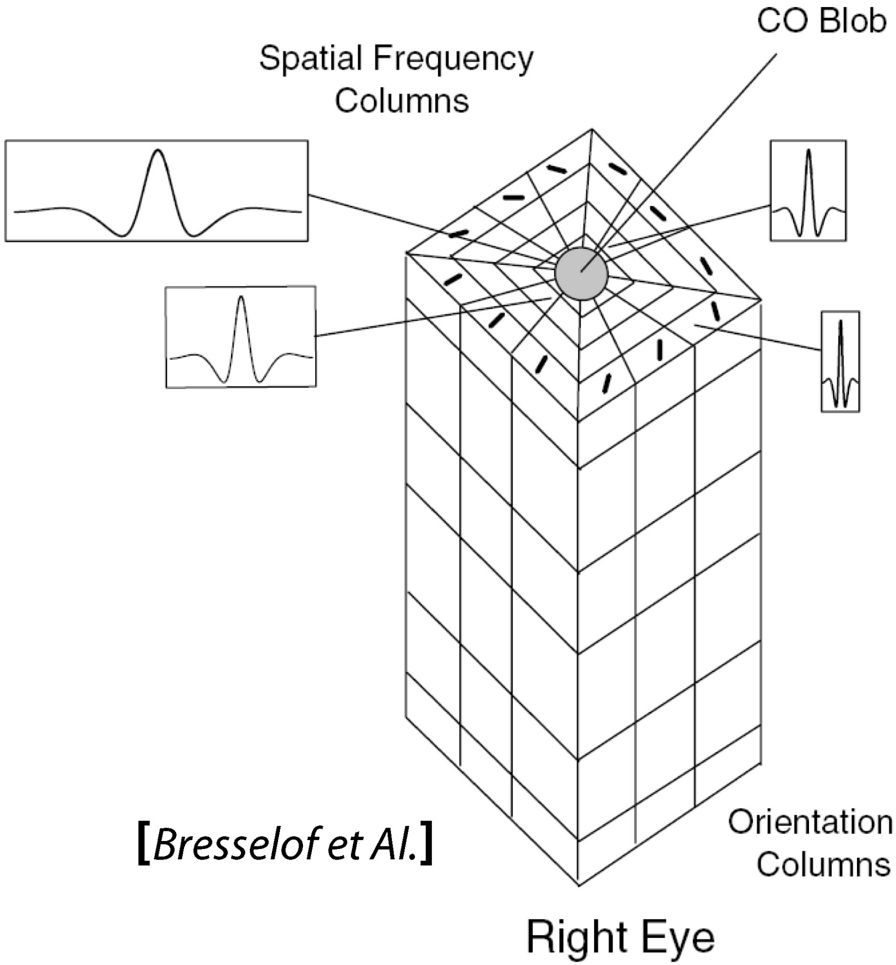

[*Bresselof et Al.*]

Contour Integration in Physiology

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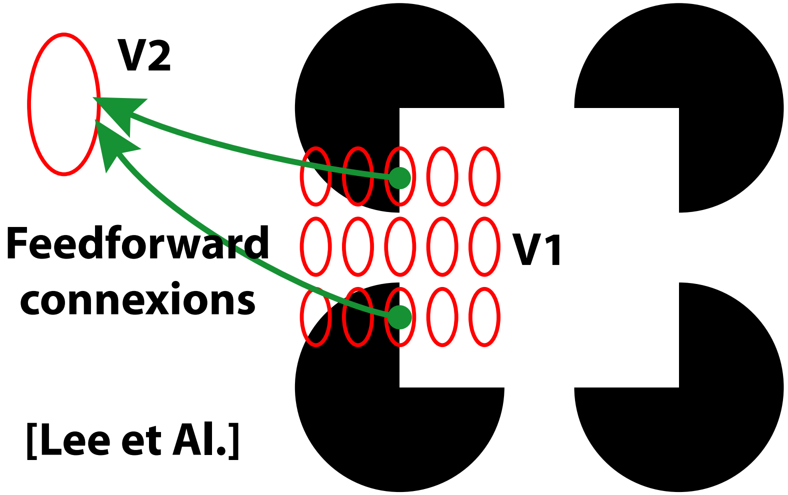
- Contour integration in V2:



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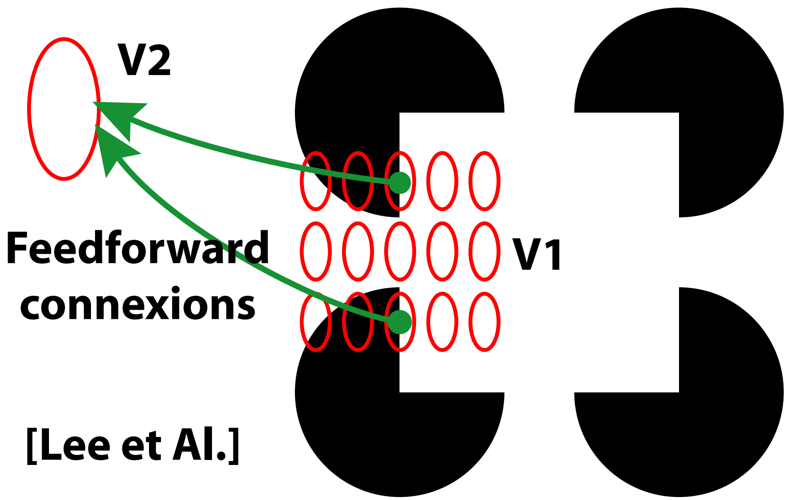

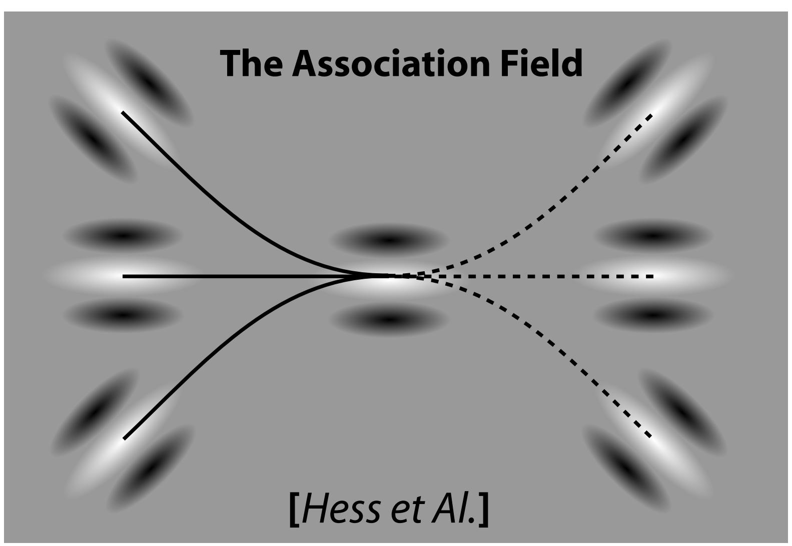

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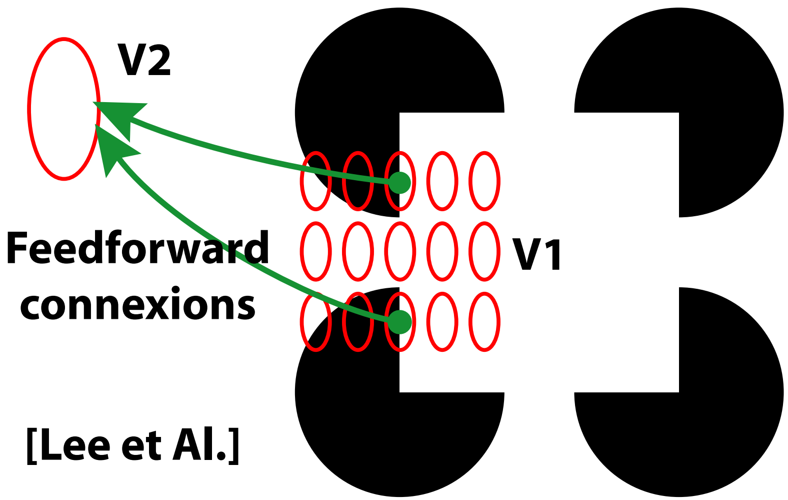

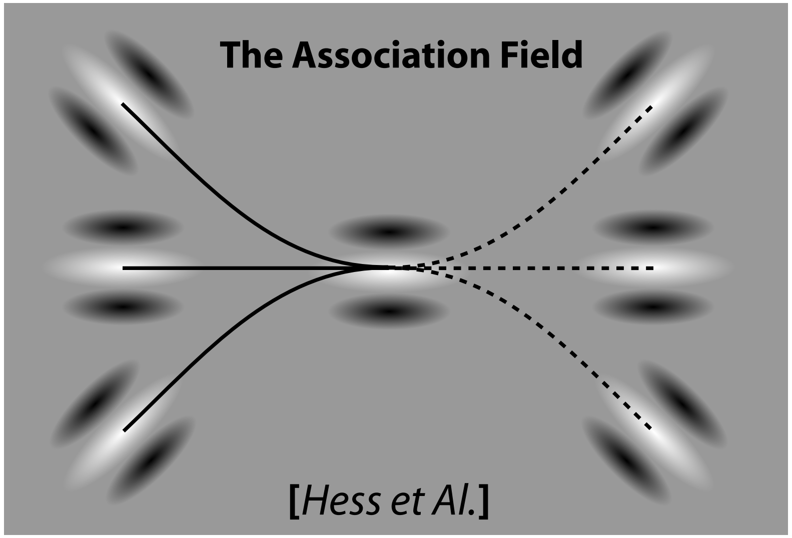

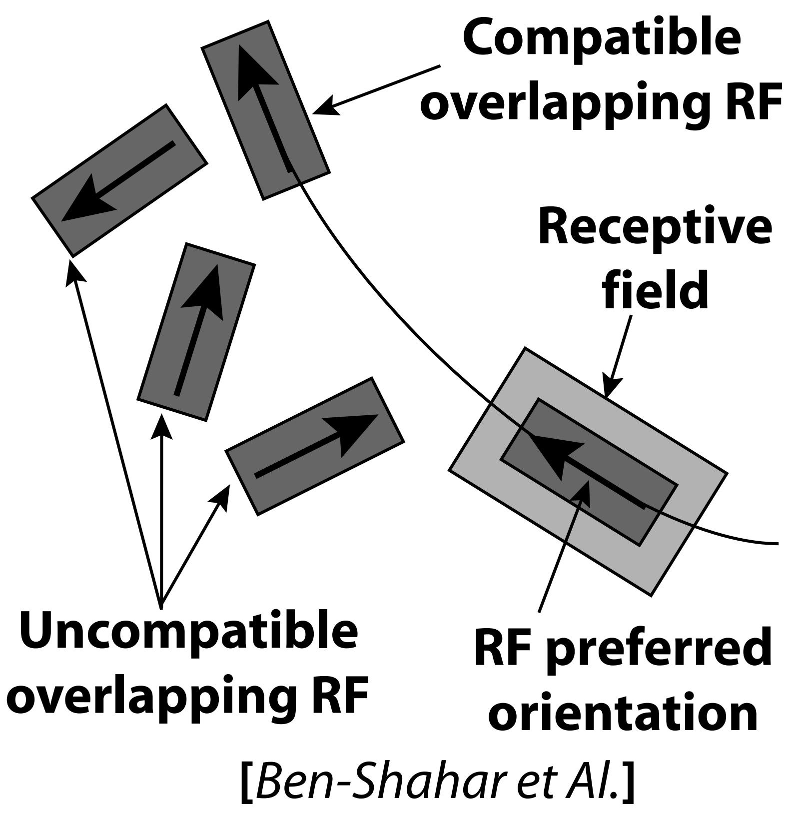

Warping the Wavelet Space

Gabriel Peyré

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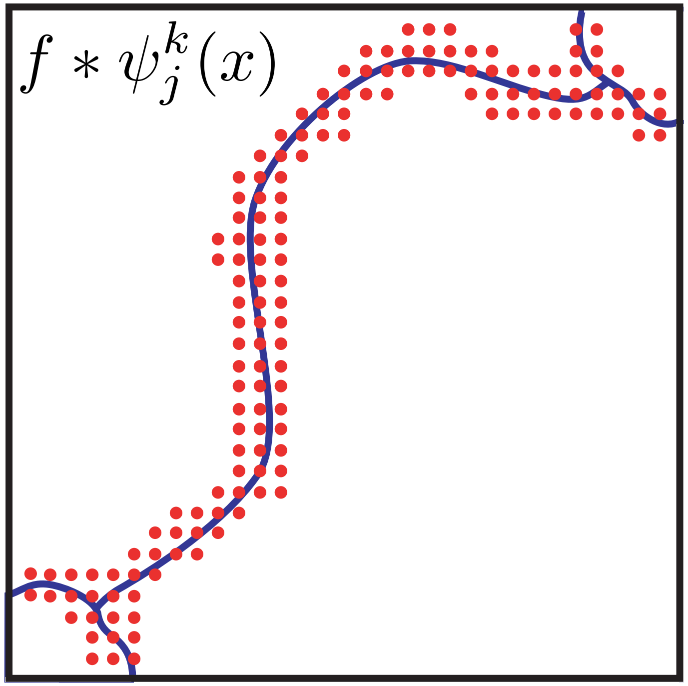
- Wavelet coefficients are samples of a regularized function:

$$\langle f\,,\,\psi^k_{j,n}\rangle = f \star \psi^k_j(2^j n) \quad \text{with} \quad \psi^k_j(x) = 2^{-j}\,\psi^k(-2^{-j}x) \quad .$$



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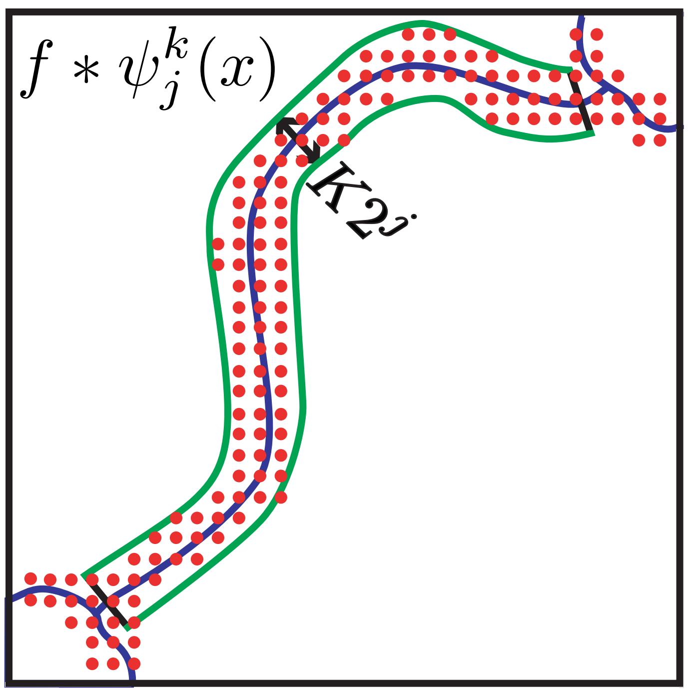

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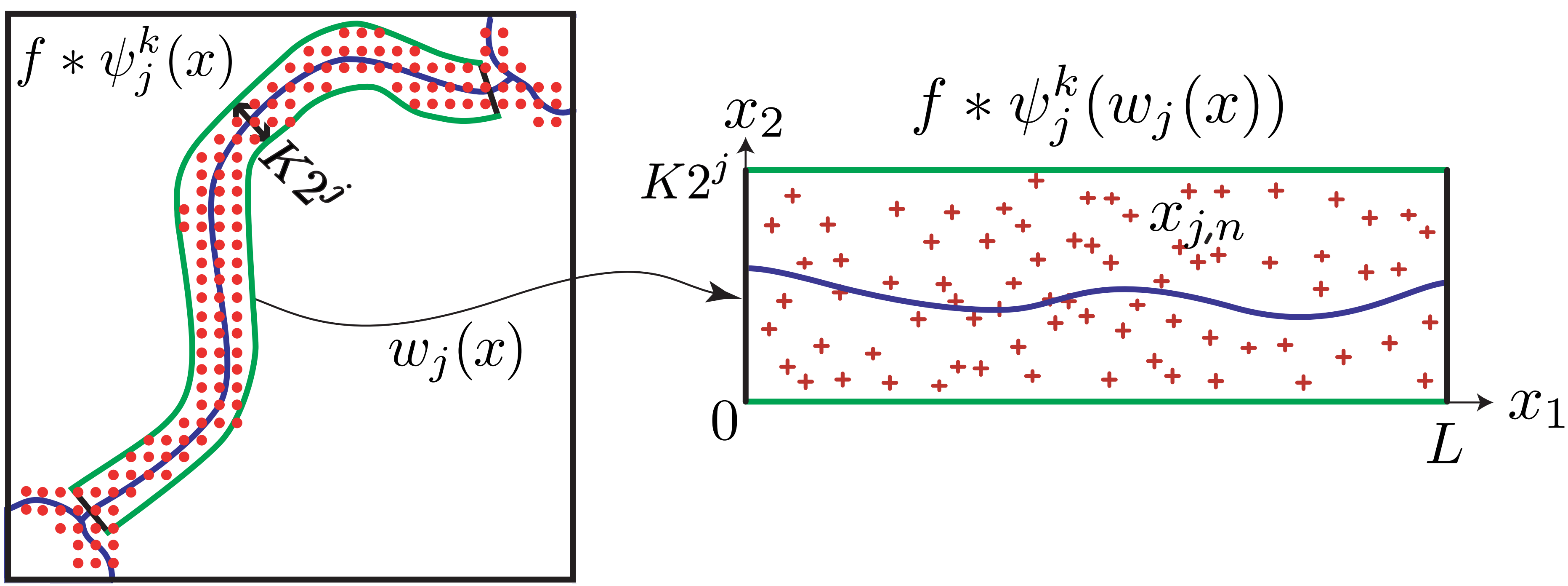

Bandeletization

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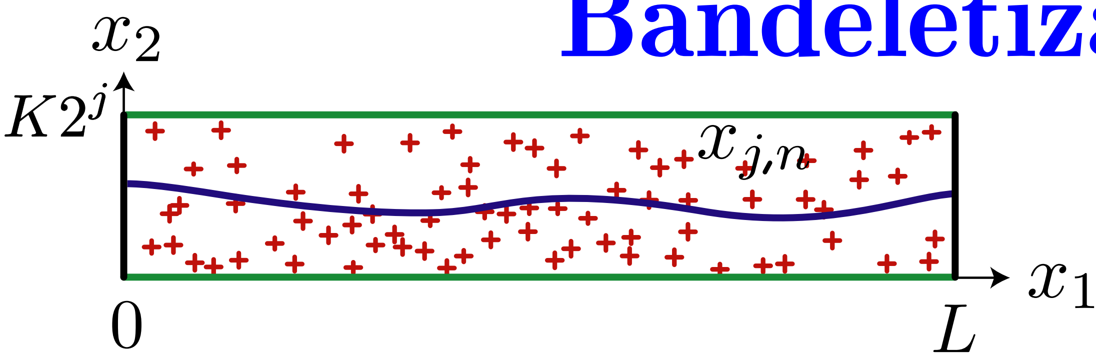
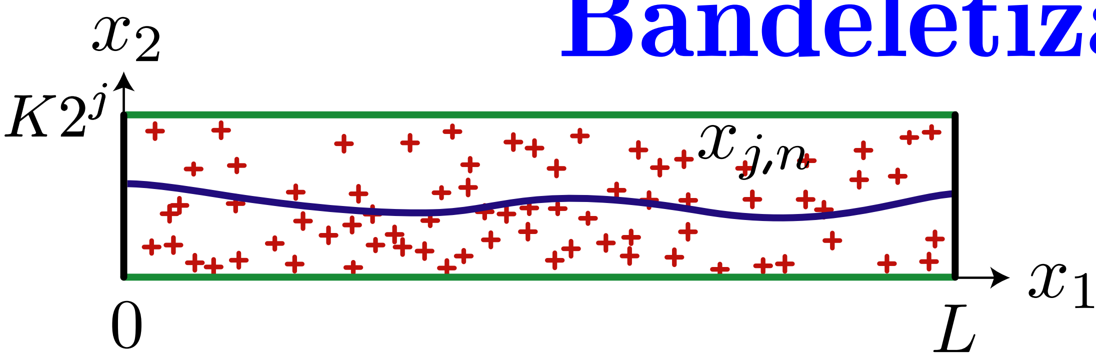


$$f_j(x) = f \star \psi_j^k(w_j(x))$$

- $$\left| \frac{\partial^{a+b} f_j(x_1, x_2)}{\partial^a x_1 \, \partial^b x_2} \right| \leqslant C \, 2^{-bj} \, 2^{-aj/\alpha} \quad .$$

Bandeletization



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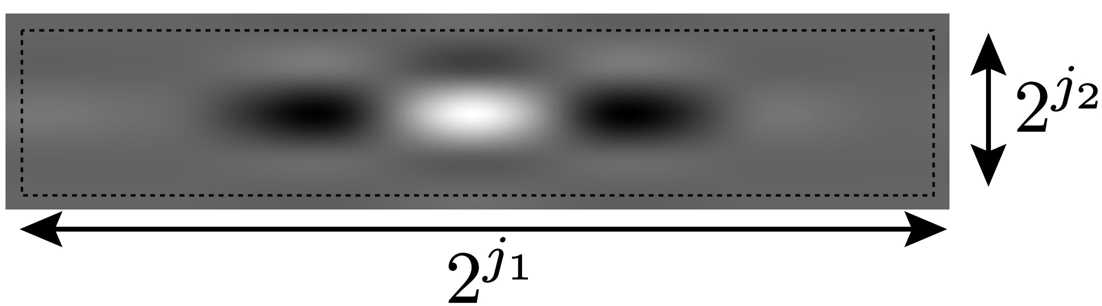
- Approximation from M wavelets of an anisotropic wavelet basis $\{\psi_{j_1,n_1}(x_1) \, \psi_{j_2,n_2}(x_2)\}_{j_1,n_1,j_2,n_2}$:

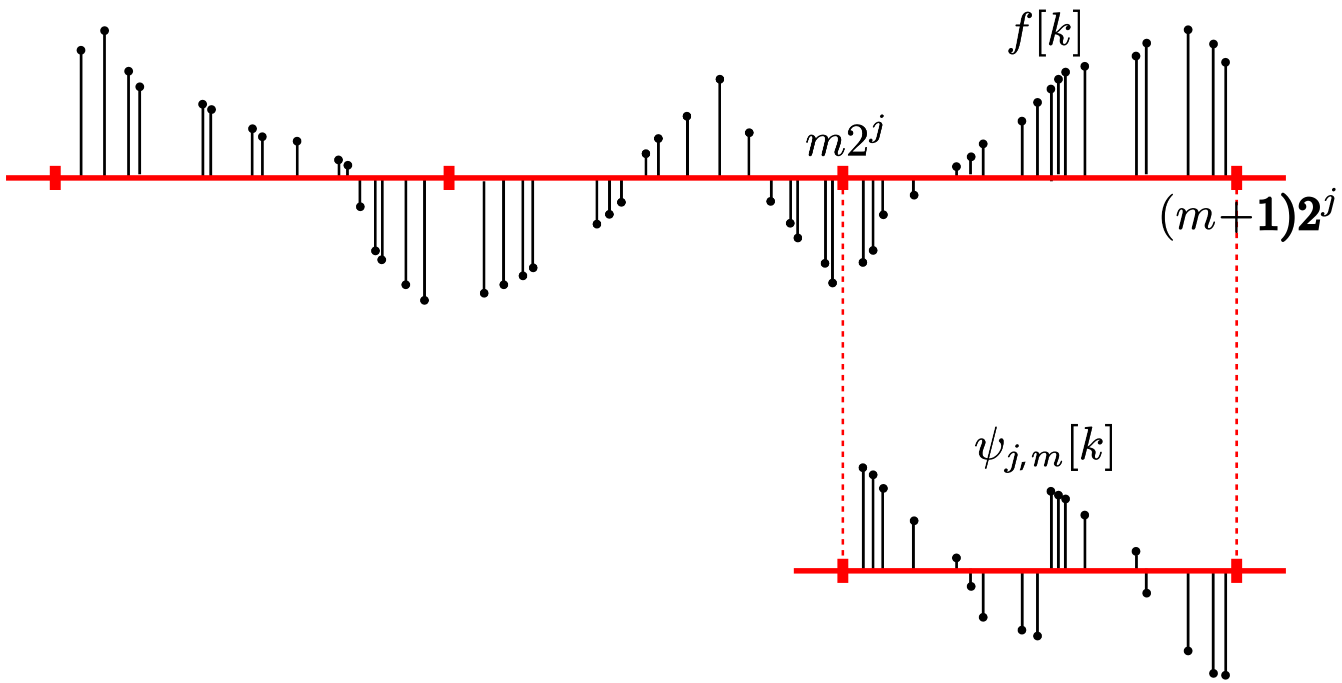


$$\|f_j - f_{j,M}\|^2 \leqslant C \, M^{-\alpha} \ .$$

Irregularly Sampled Alpert Multiwavelets

Irregularly Sampled Alpert Multiwavelets

- Alpert discontinuous polynomial multiresolution approximation:
$\mathbf{V}_j = \{ f \ : \ f \text{ is a polynomial of degree p on } [m2^j, (m+1)2^j) \}$



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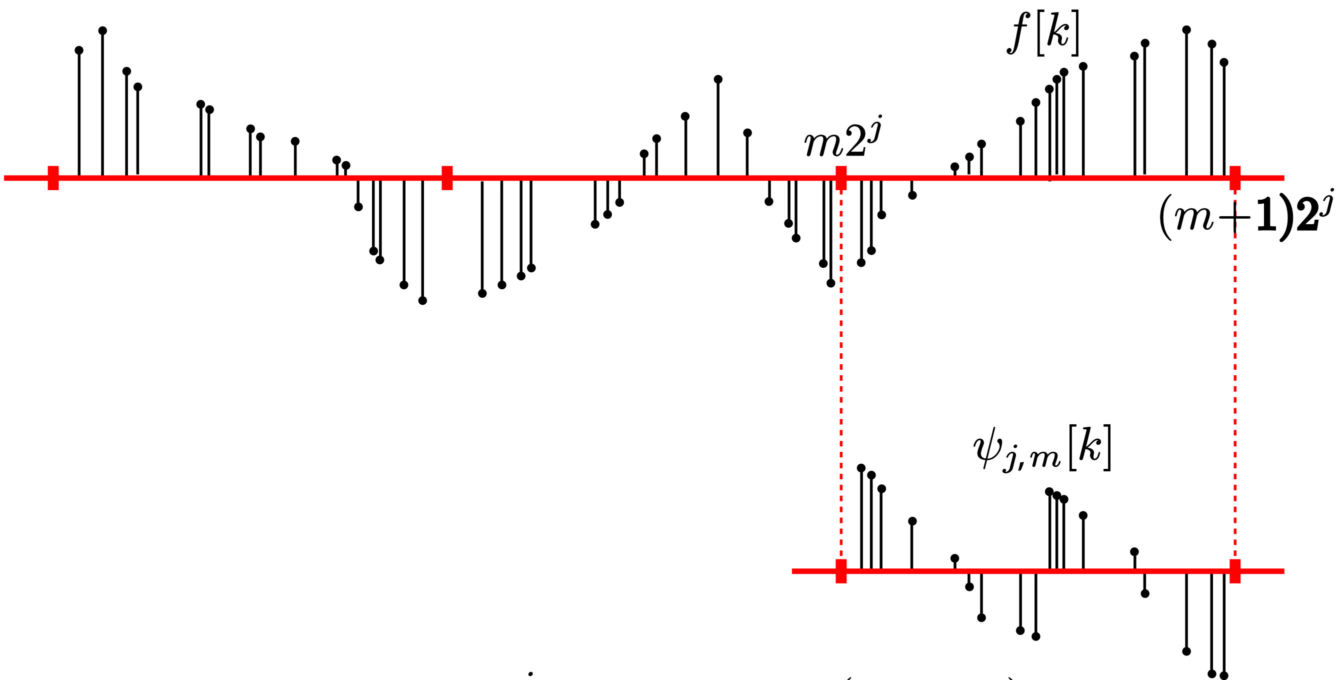

- On each interval of size 2^j there are $(p+1)$ wavelets having $(p+1)$ vanishing moments.

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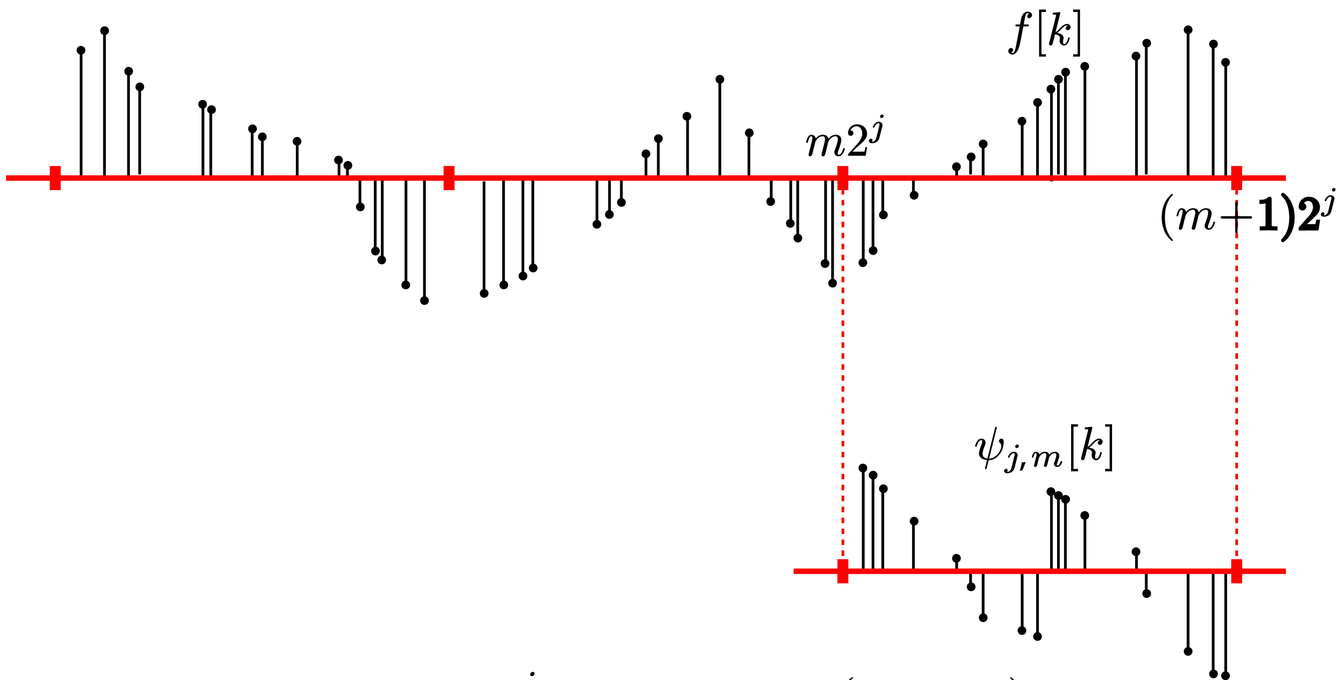

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- Alpert fast wavelet transform is $O(N)$ for N irregularly spaced samples.

2D Discrete Alpert Multiwavelets

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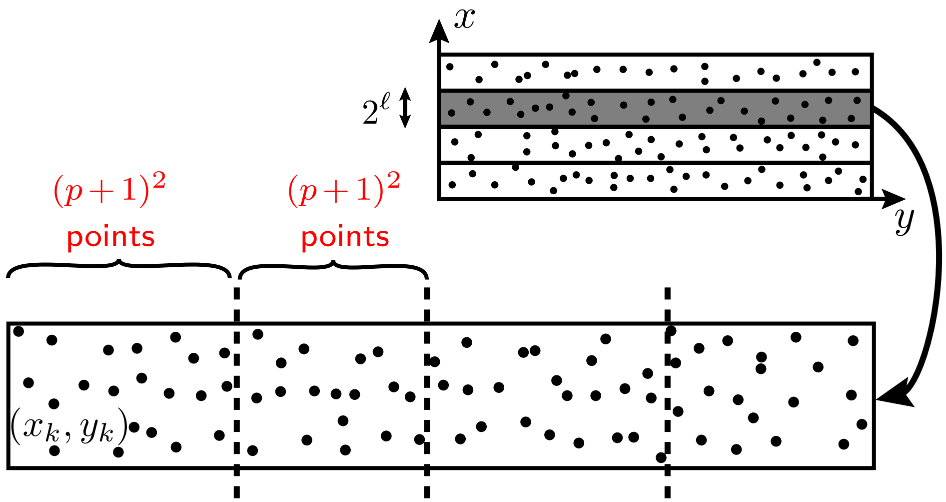
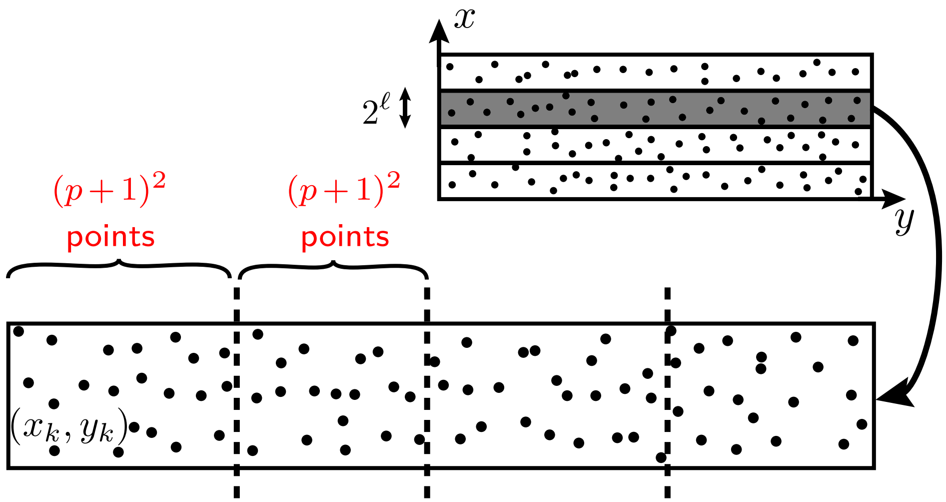


- On each slice take basis vectors $(x_k^i y_k^j)$ for $i, j = 0 \dots p$.

2D Discrete Alpert Multiwavelets



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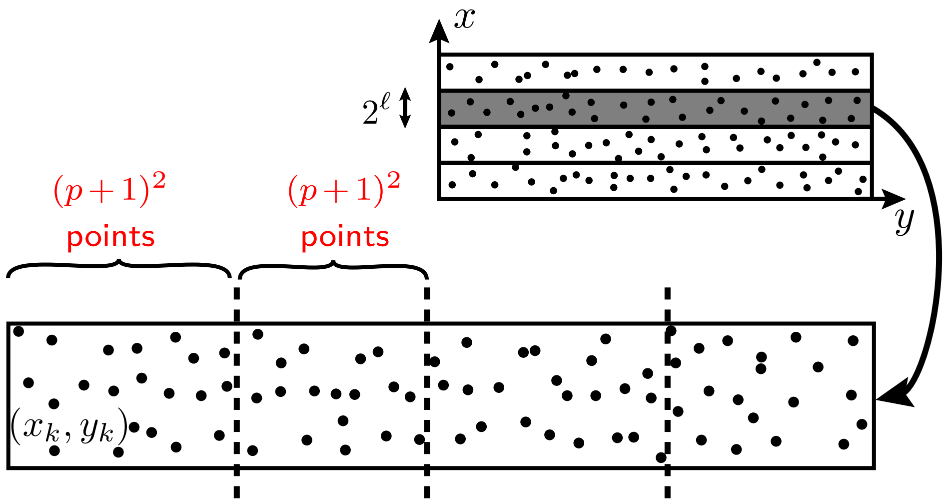

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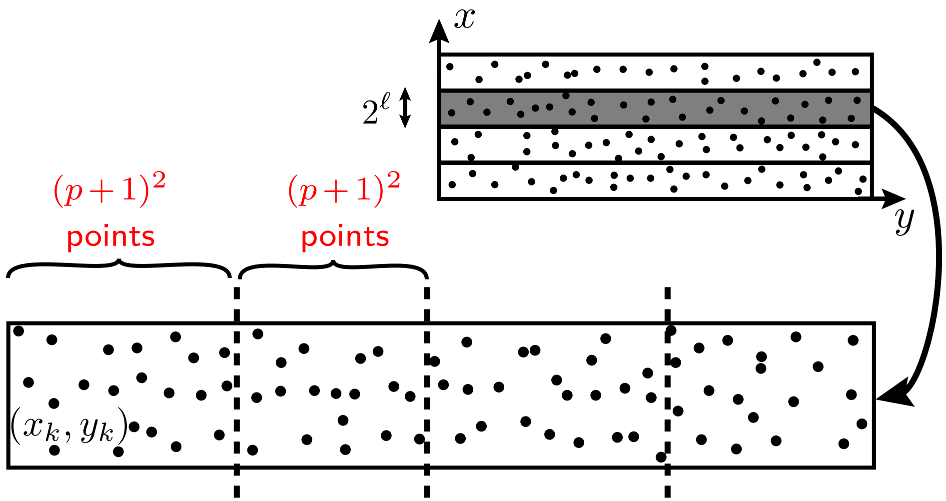

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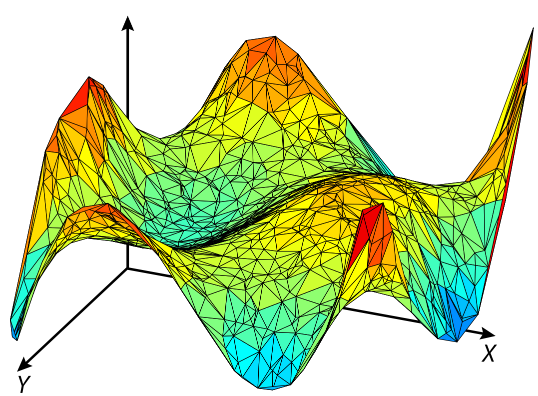
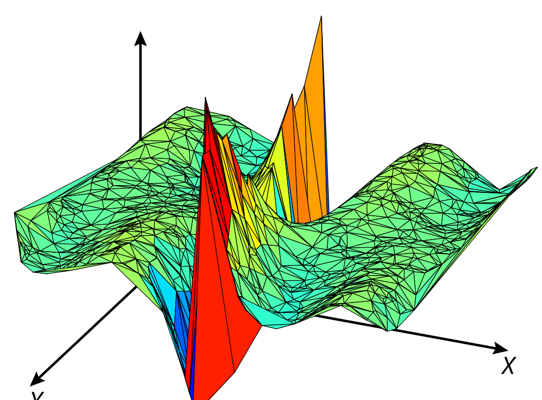


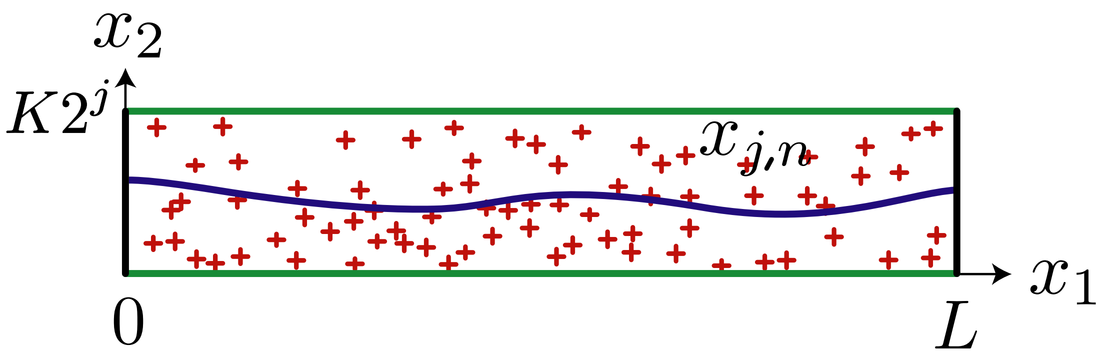
Scaling function



Wavelet function

Bandeletization with 2D Alpert Wavelets

Bandeletization with 2D Alpert Wavelets

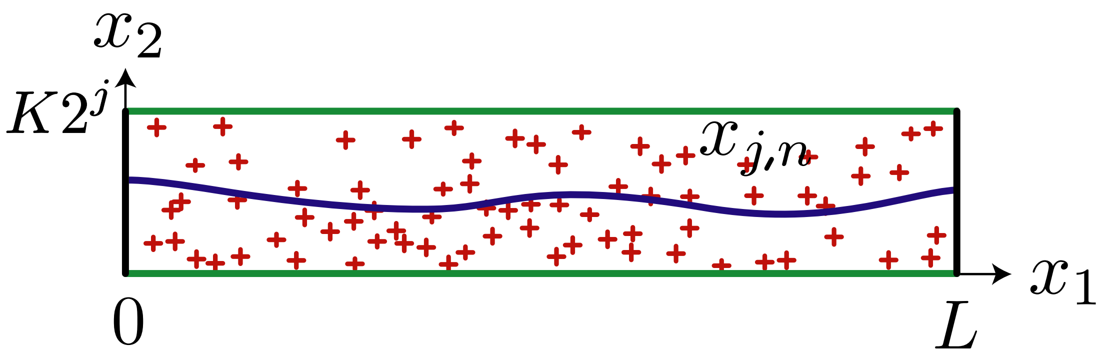


$$\bar{f}_j[n] = f_j(x_{j,n}) \in \mathbb{R}^{N_j}$$

- Approximation of $\bar{f}_j[n]$ in a 2D anisotropic Alpert wavelet basis $\{a_{j,m}[n]\}_{0 \leqslant n < N_j}$:

$$\bar{f}_{j,M}[n] = \sum_{|\langle \bar{f}_j, a_m \rangle| > T_M} \langle \bar{f}_j, a_{j,m} \rangle \, a_{j,m}[n] \; .$$

Bandeletization with 2D Alpert Wavelets



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- Requires $O(N_j)$ operations and

$$\|\bar{f}_j - \bar{f}_{j,M}\|^2 \leqslant C \, M^{-\alpha} \; .$$

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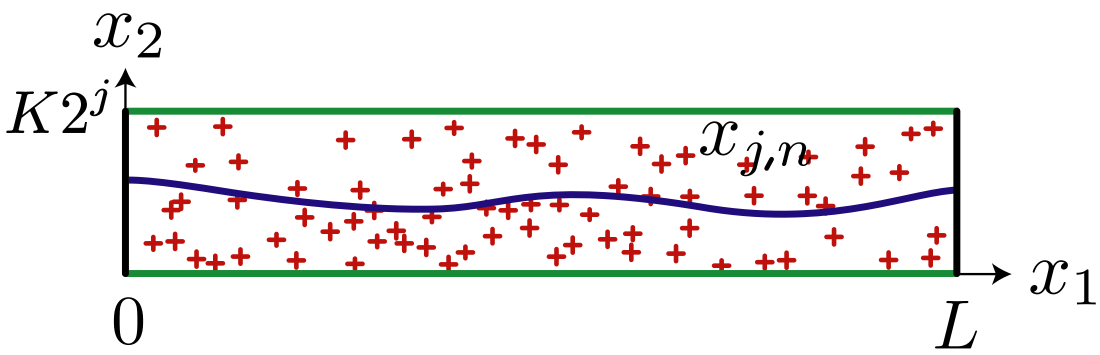

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- Requires $O(N_j)$ operations and

$$\|\bar{f}_j - \bar{f}_{j,M}\|^2 \leqslant C \, M^{-\alpha} \ .$$

- Similar to V2 neurons.

Second Generation Bandelets

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- $$b^k_{j,m}(x) = \sum_{n=1}^{N_j} a_{j,m}[n]\, \psi^k_{j,n}(x)\ .$$

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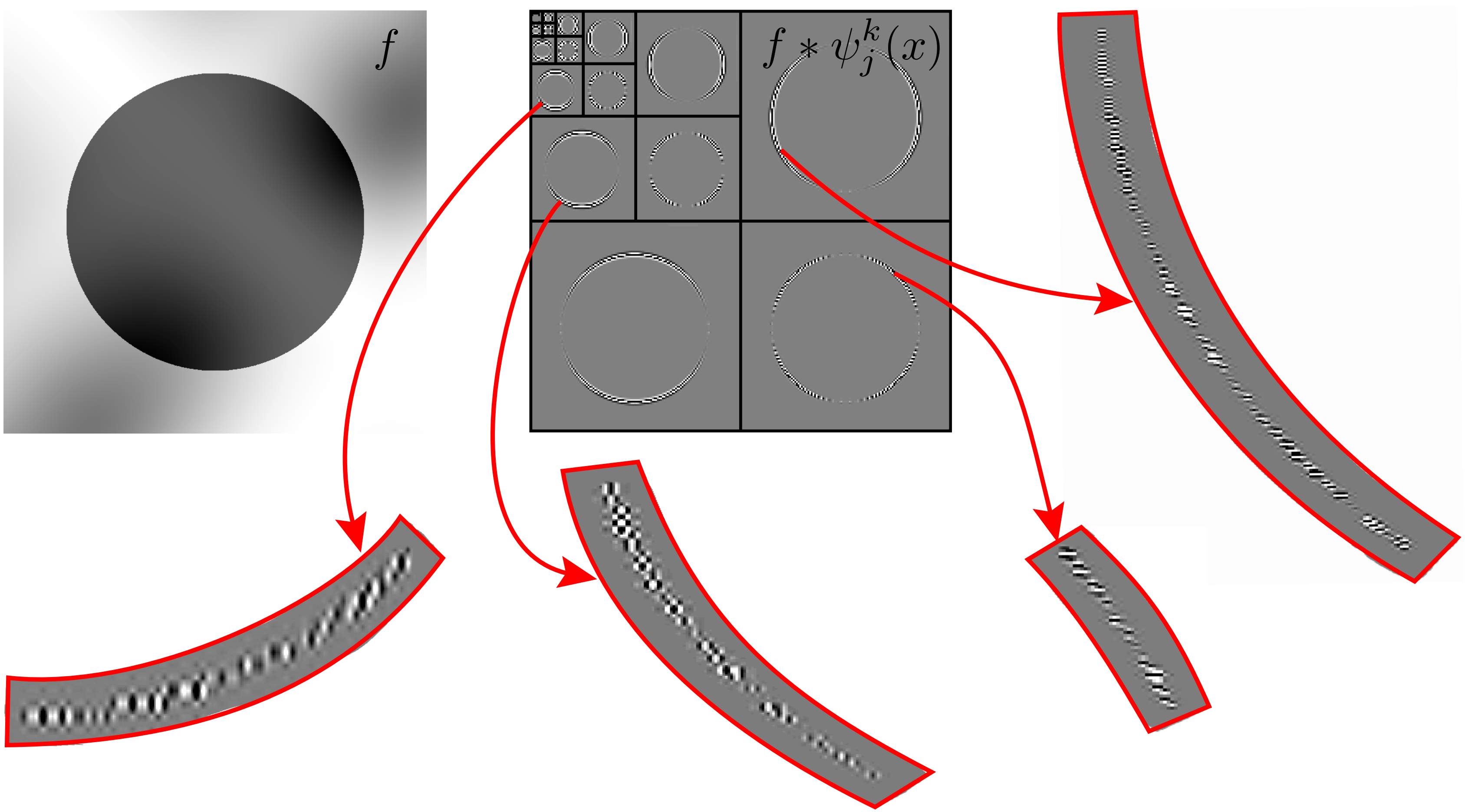
Second Generation Bandelets

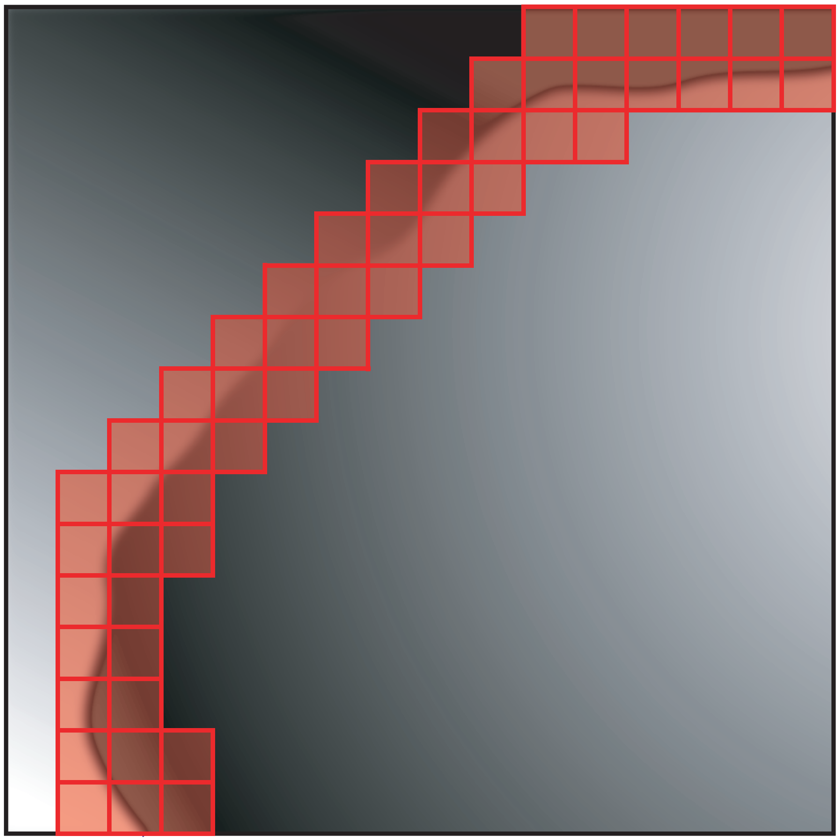
- $$b^k_{j,m}(x) = \sum_{n=1}^{N_j} a_{j,m}[n]\,\psi^k_{j,n}(x)\ .$$
- Bandelet orthonormal basis: $\left\{\psi^k_{j,n}\right\}_{k,j,n} \cup \left\{b^k_{j,m}\right\}_{k,j,m}$.

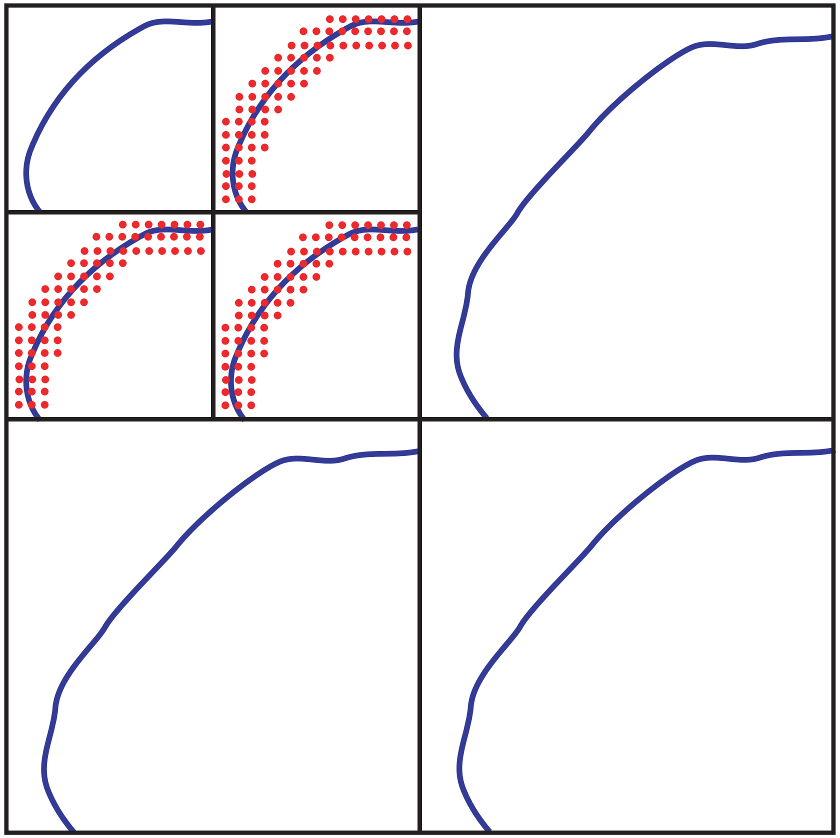


Multiscale Geometry

Multiscale Geometry

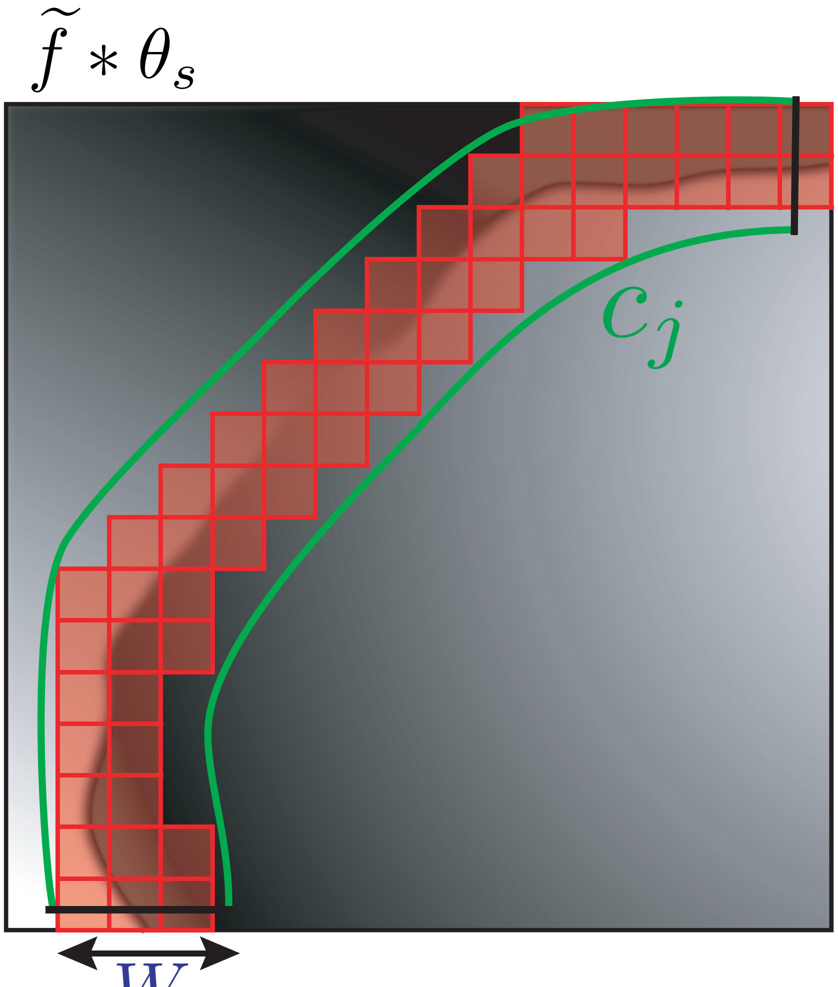
$\widetilde{f} * \theta_s$

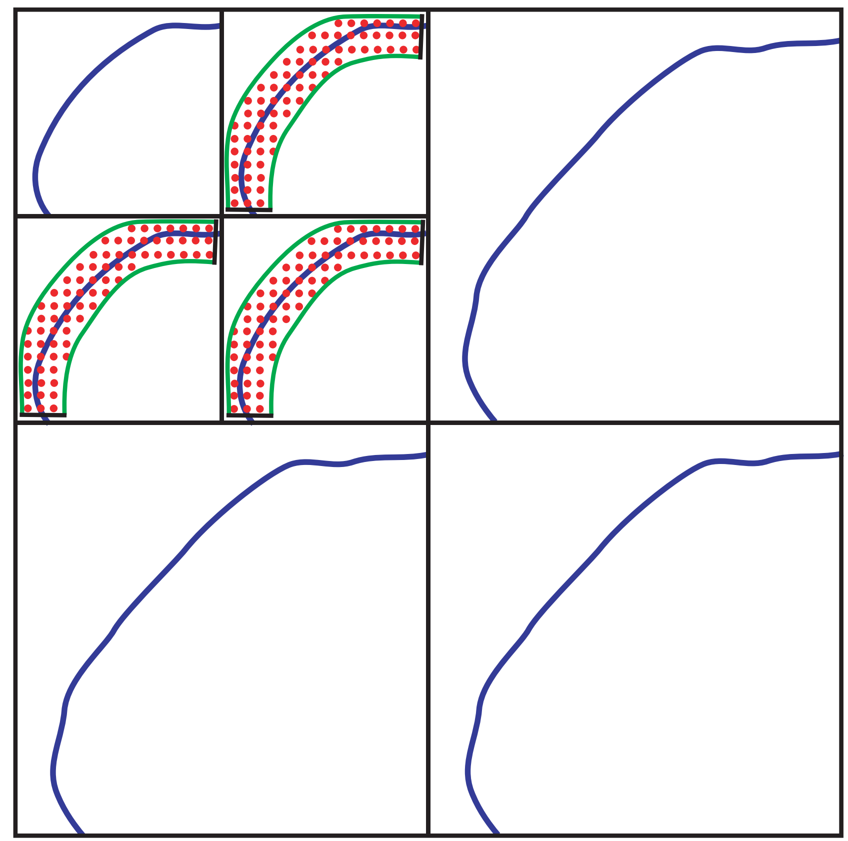




- detection threshold D_j

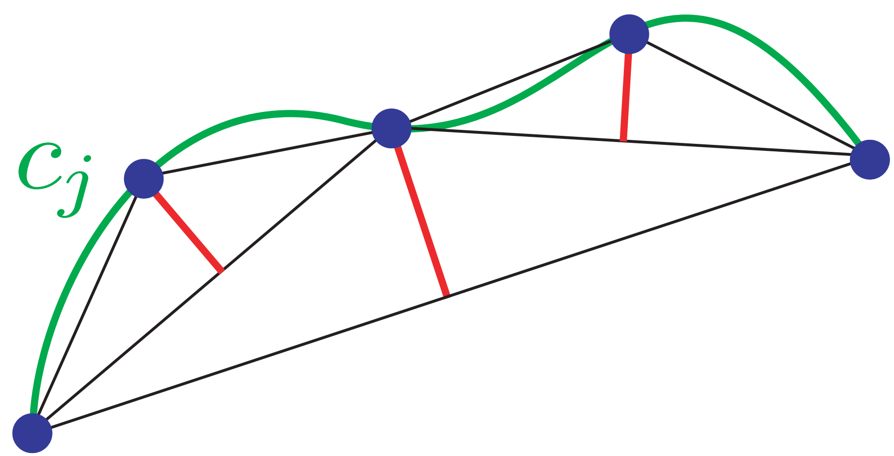
Multiscale Geometry



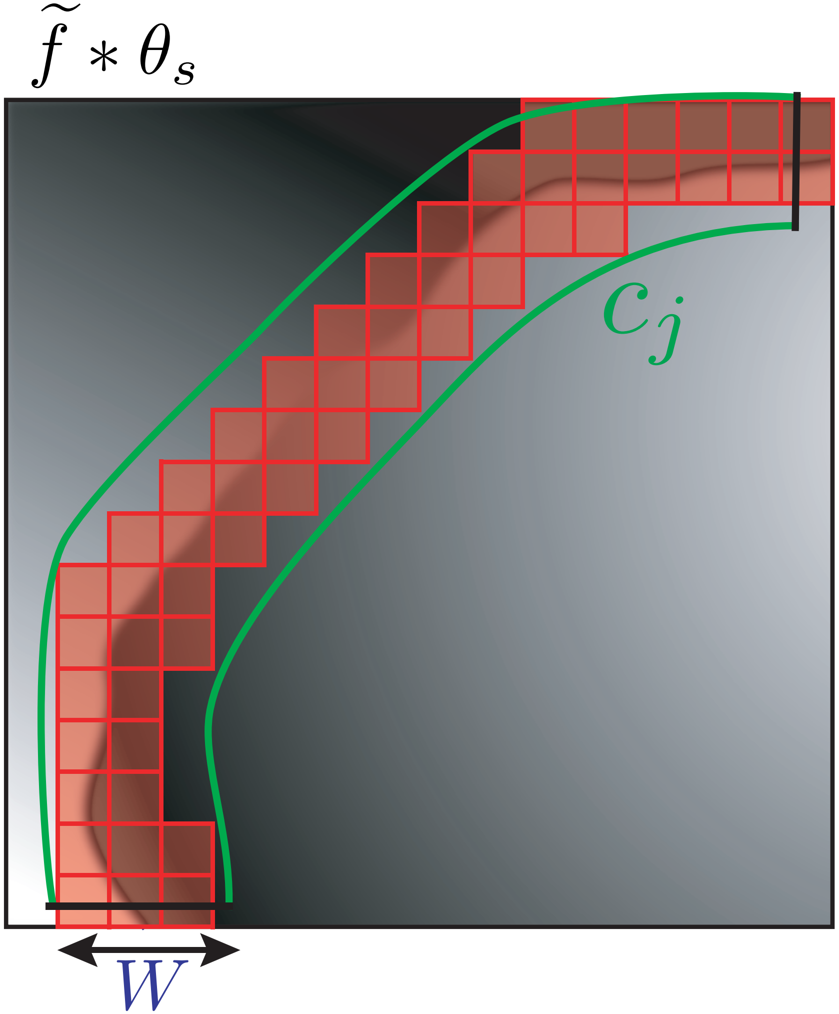


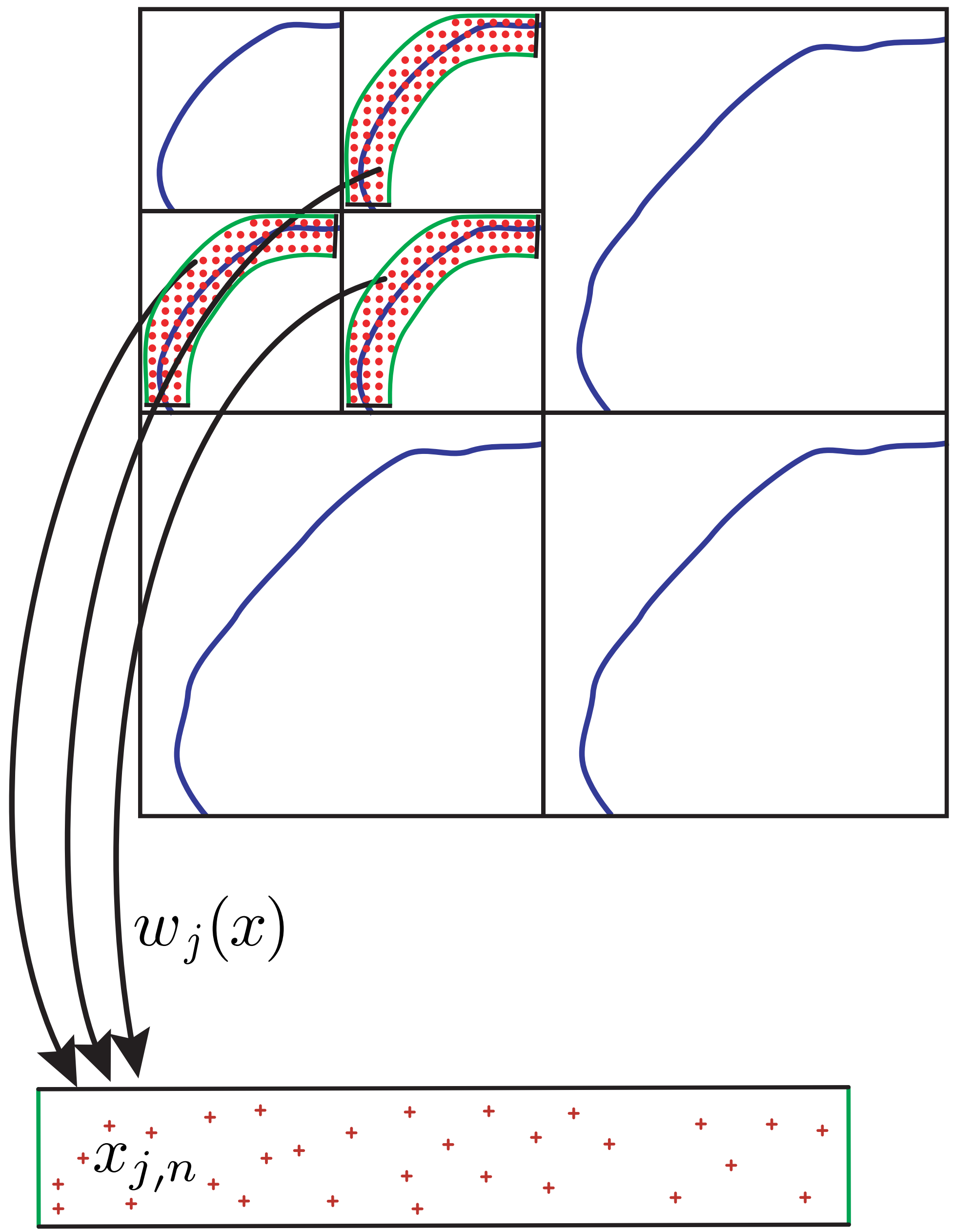
- wavelet coefficients are in a band of width $W = \max(2^j K, s)$
- detection threshold D_j
- c_j is parameterized with a normal subdivision

[*Daubechies, Runborg, Sweldens*]



Multiscale Geometry





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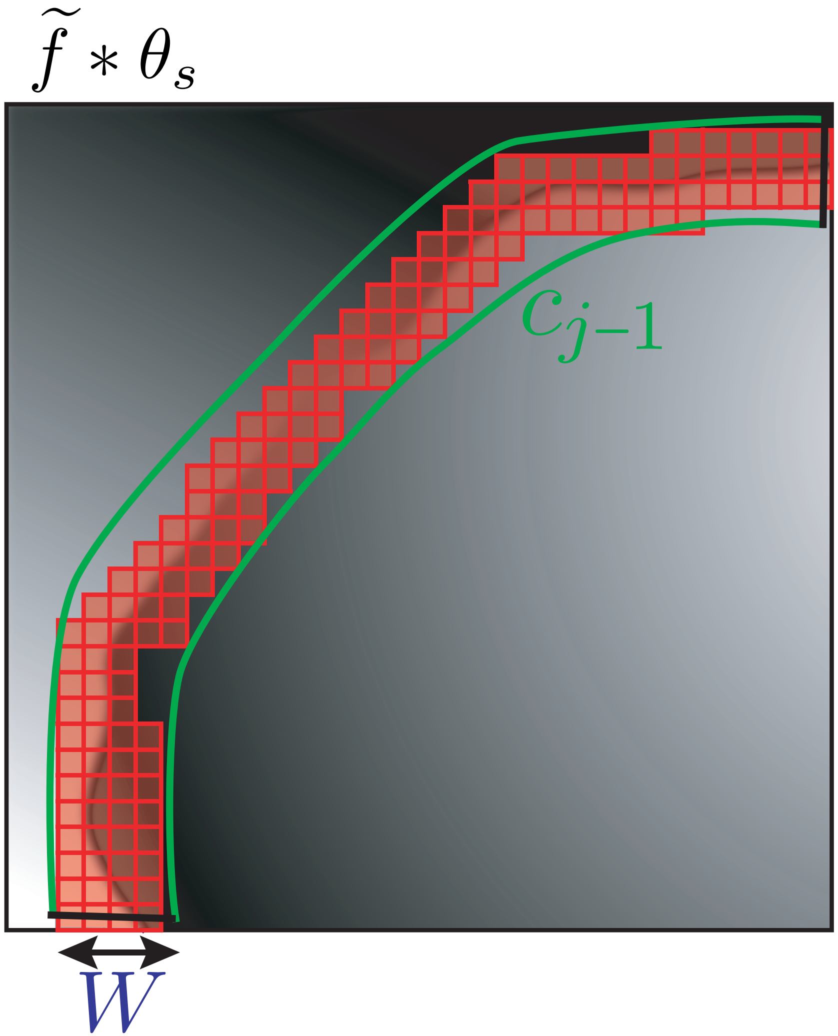
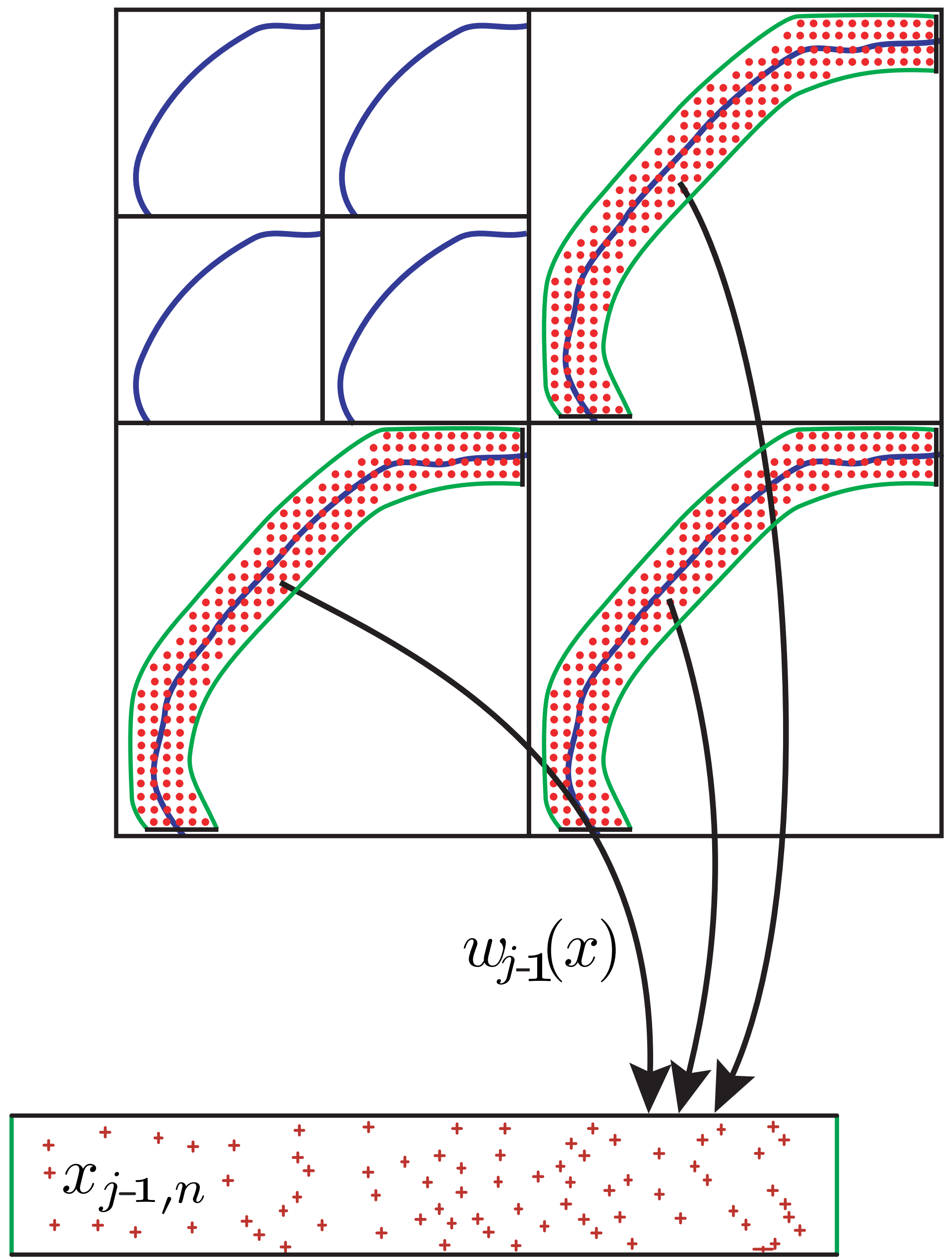
Multiscale Geometry



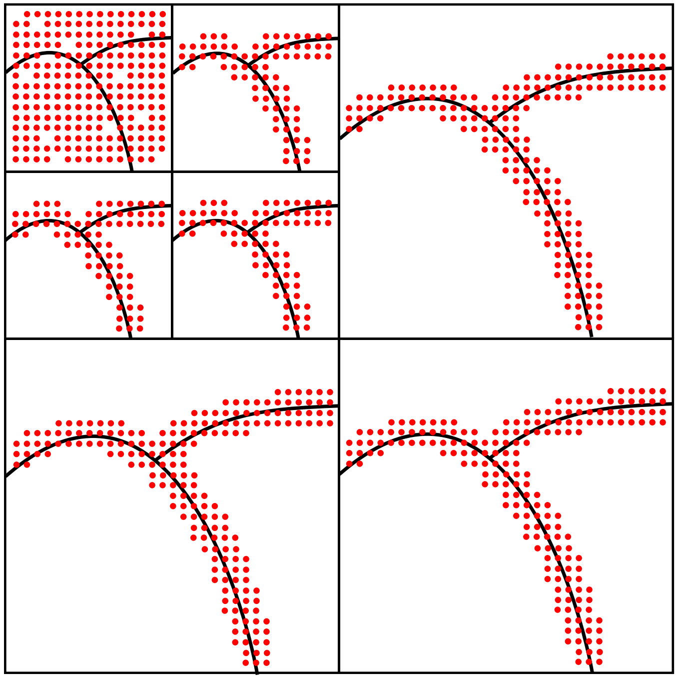


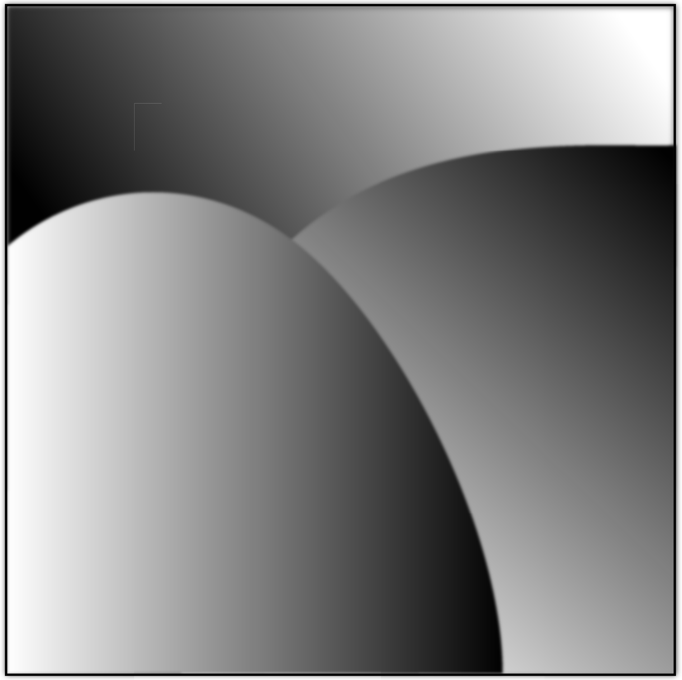
- wavelet coefficients are in a band of width $W = \max(2^{j-1}K, s)$
- detection threshold D_{j-1}
- $c_{j\text{-}1}$ is parameterized with a normal subdivision

[*Daubechies, Runborg, Sweldens*]

Dyadic Segmentation

Dyadic Segmentation





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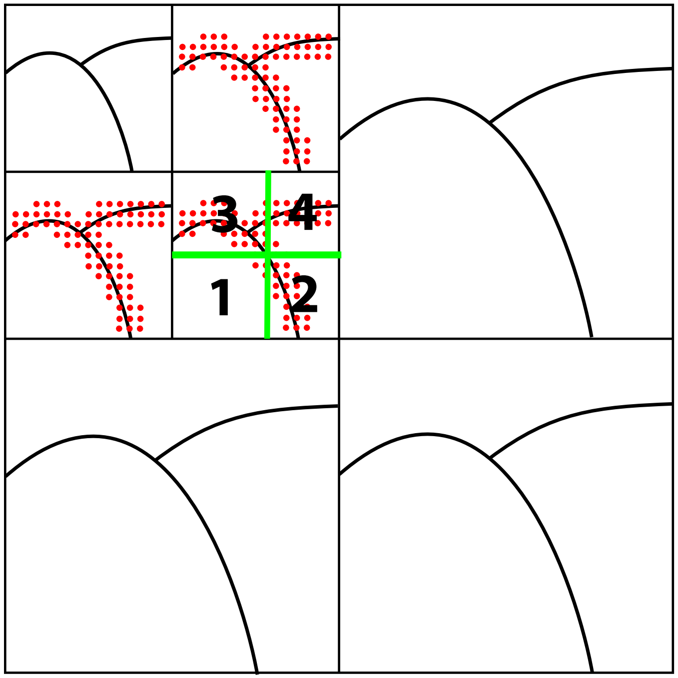

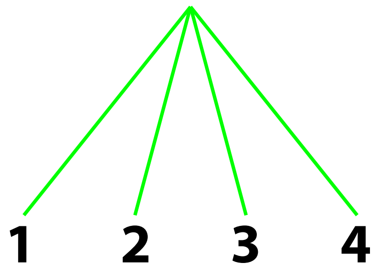

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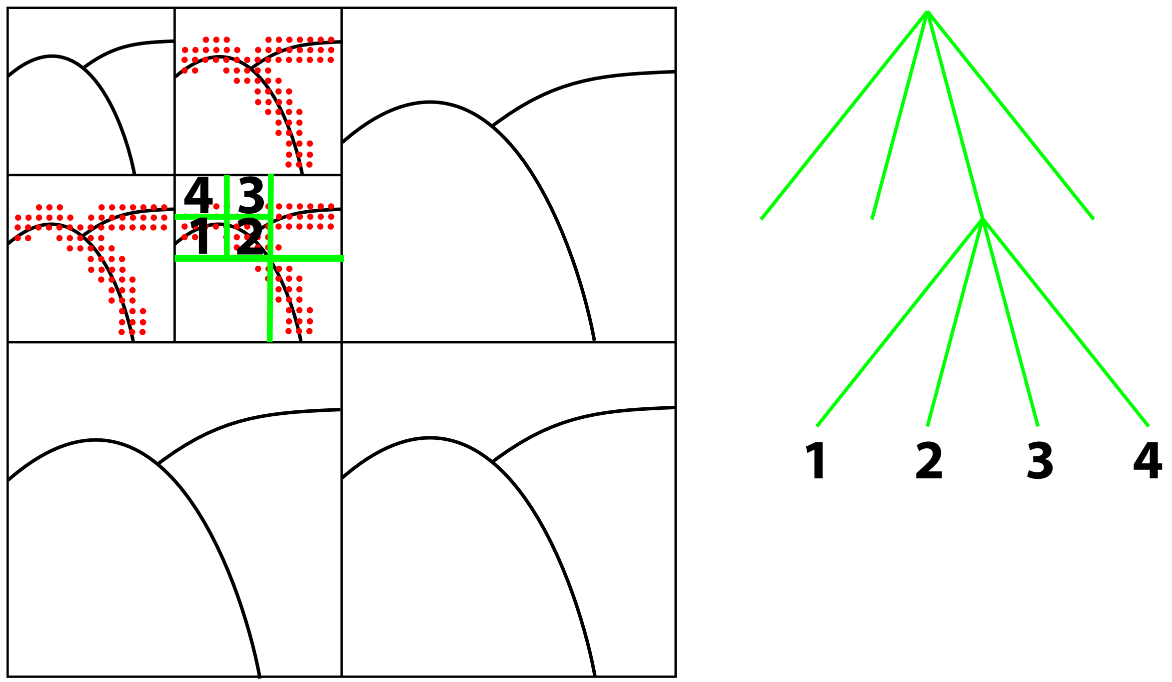

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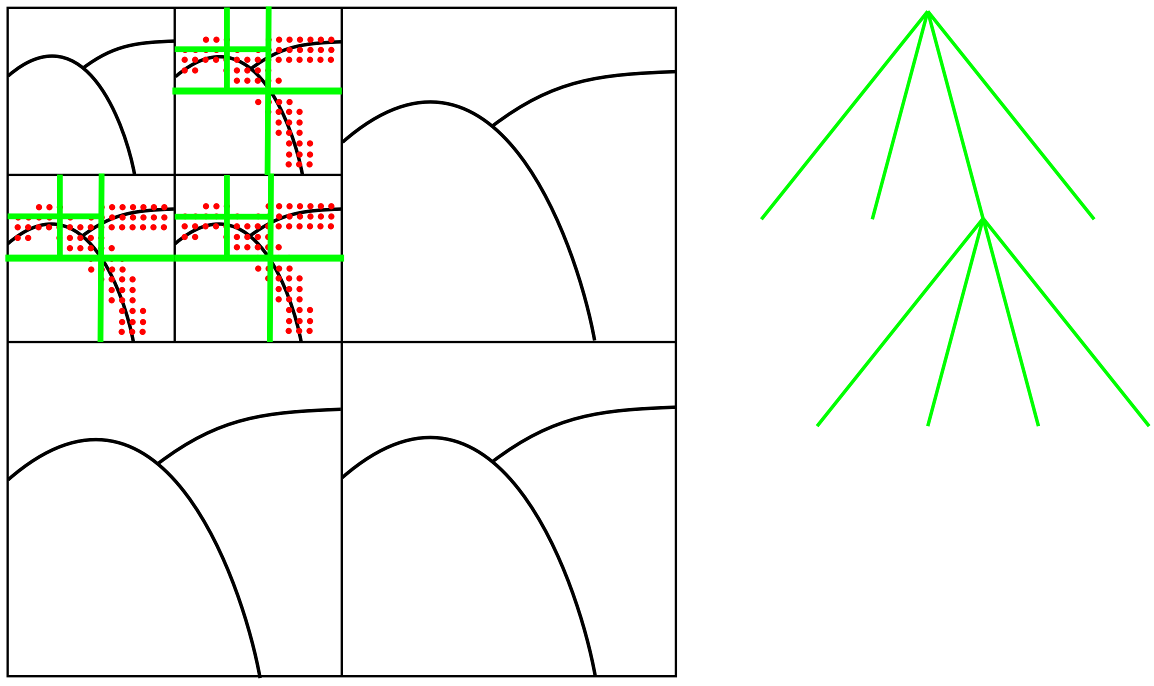

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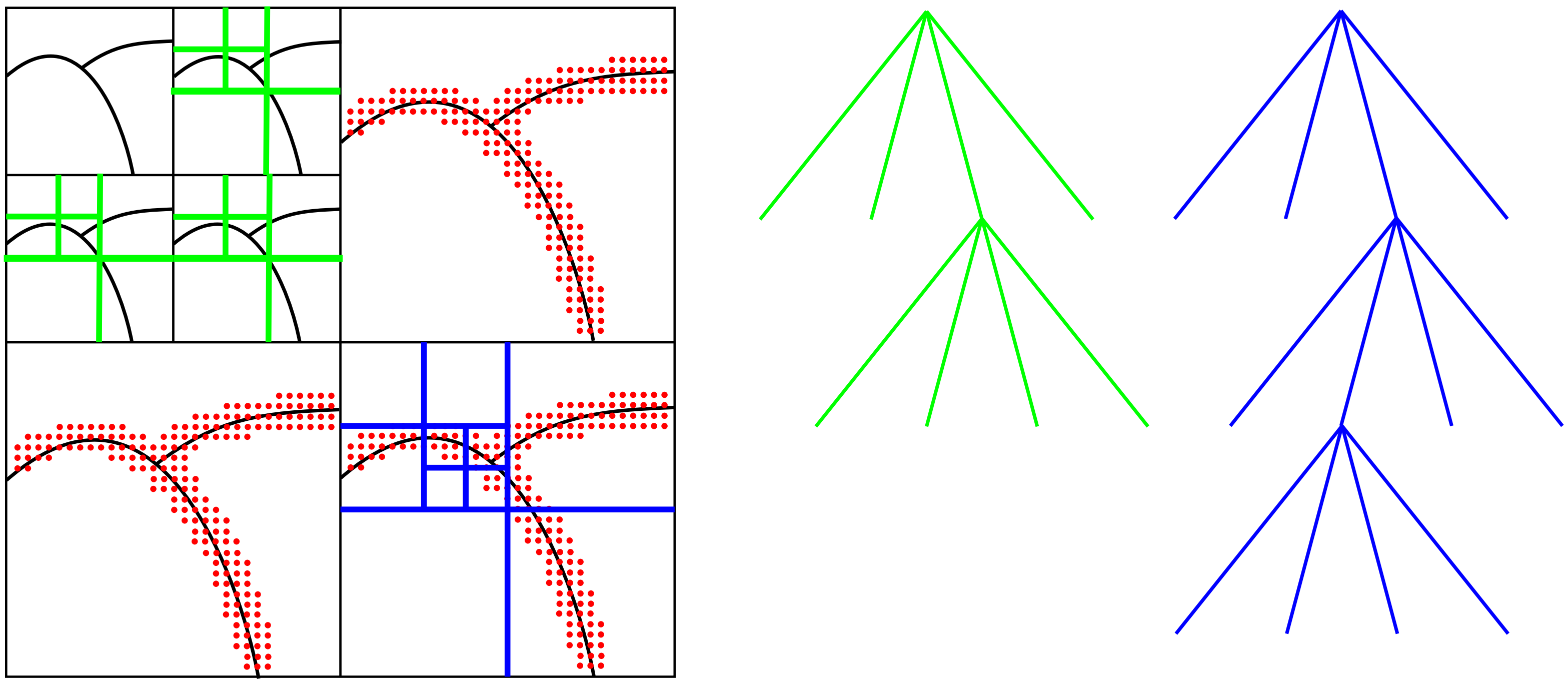

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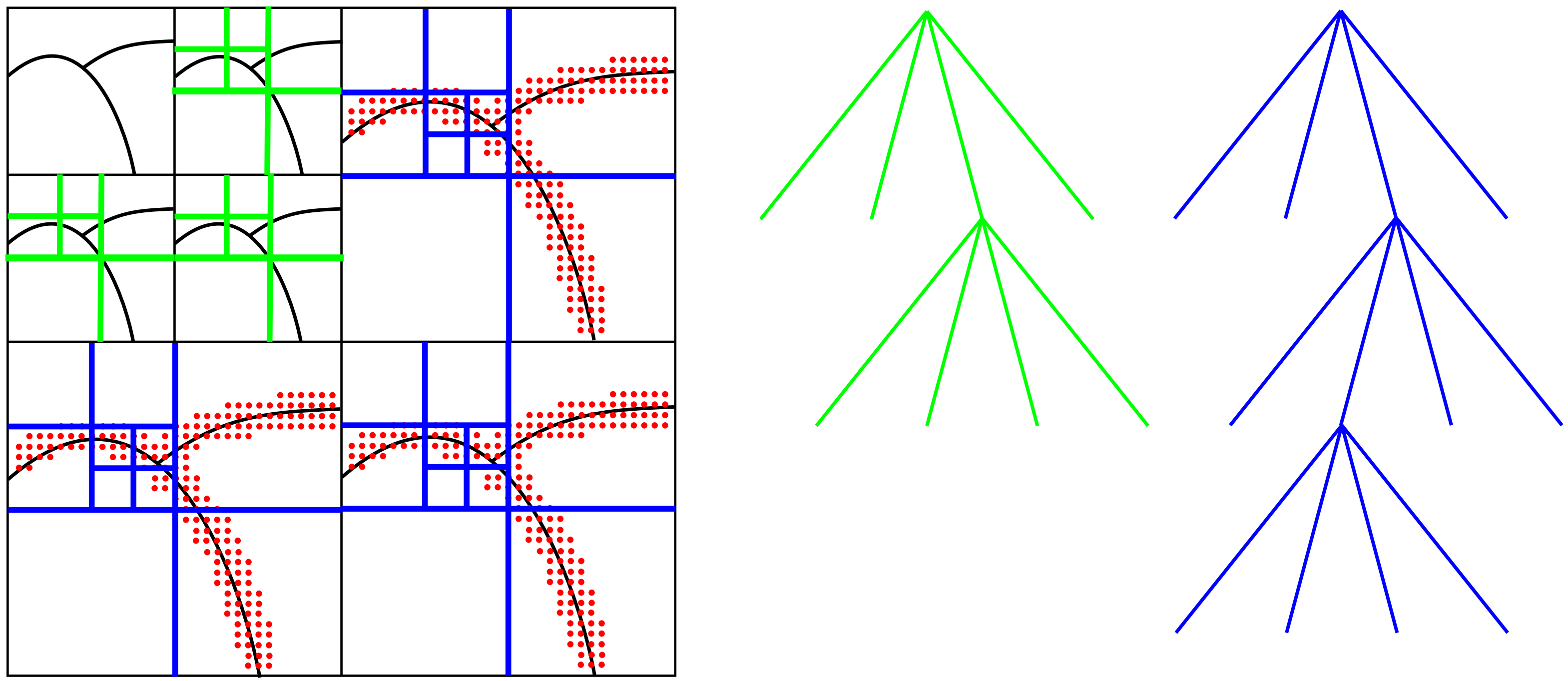

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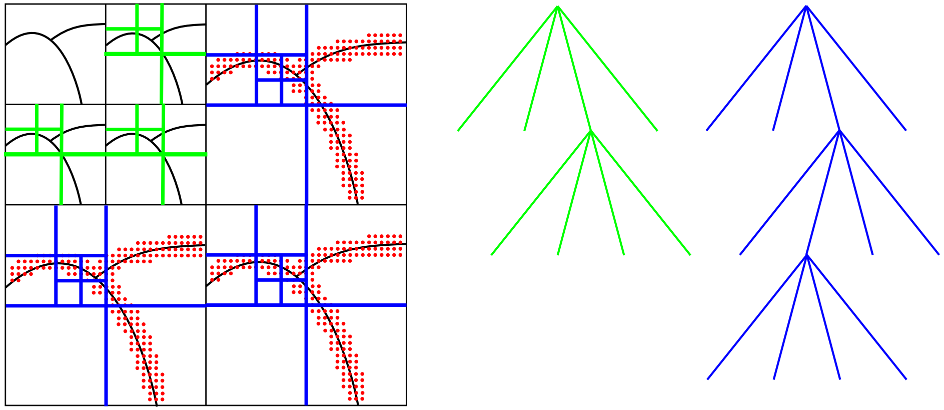
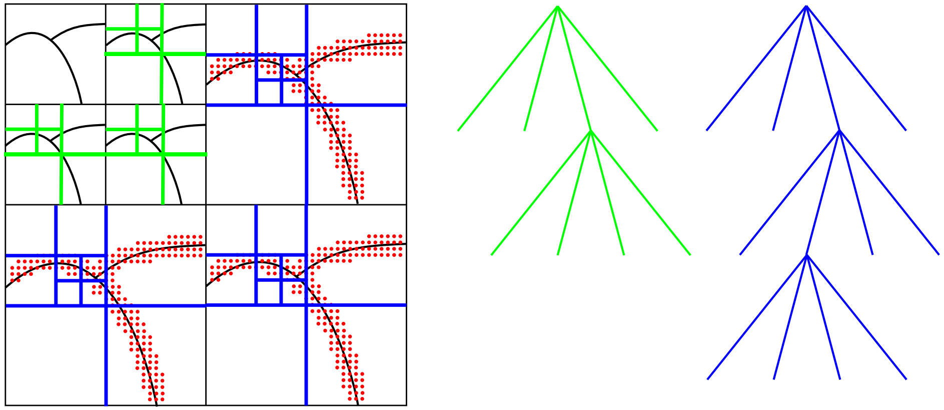


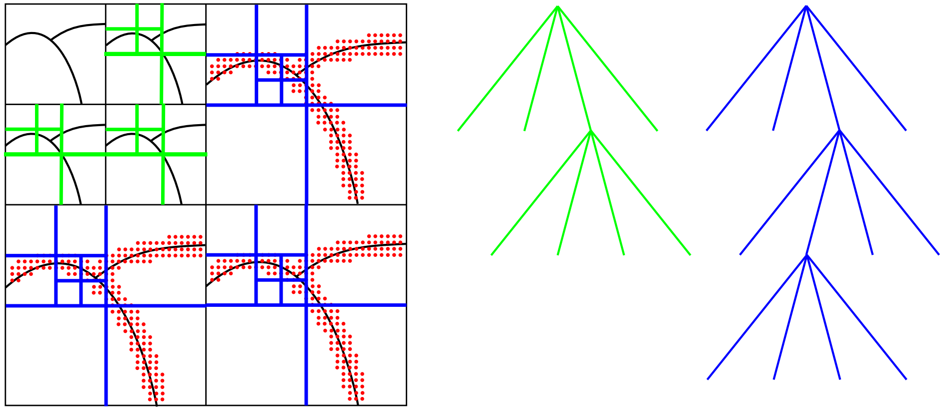
Dyadic Segmentation



- Total number of bandelet, wavelet and geometric coefficients:

$$M = \sum_j M_j = \sum_j \Big(M_{B_j} + M_{W_j} + M_{G_j} \Big)$$

Dyadic Segmentation



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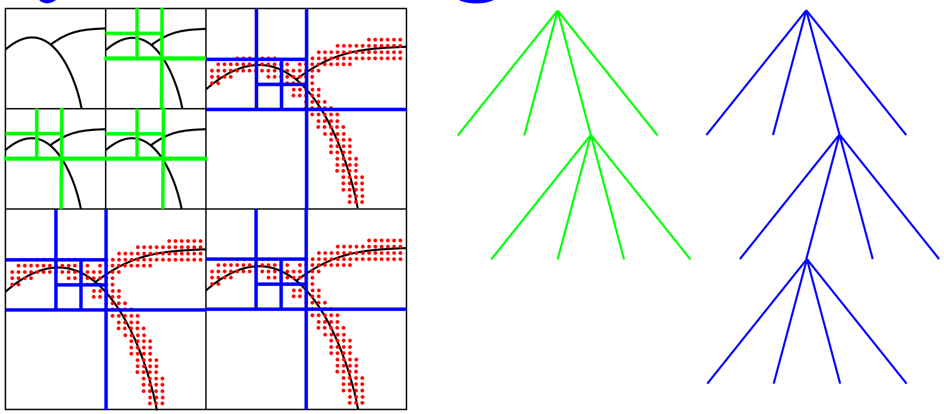

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Bandelet Representation

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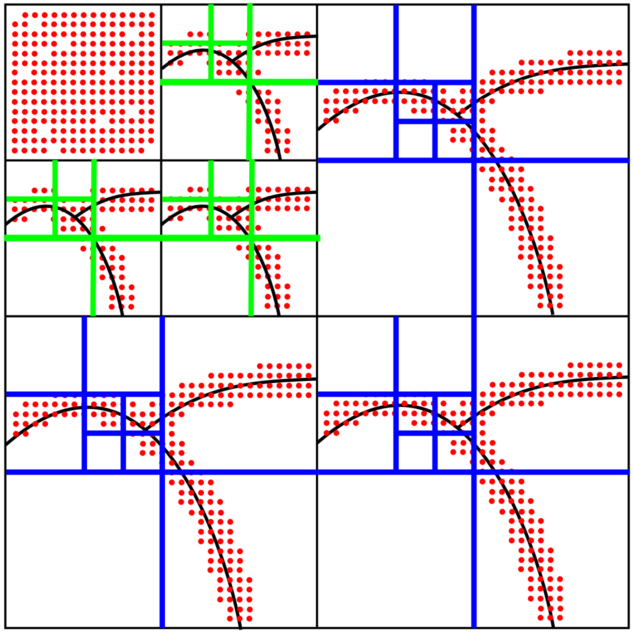

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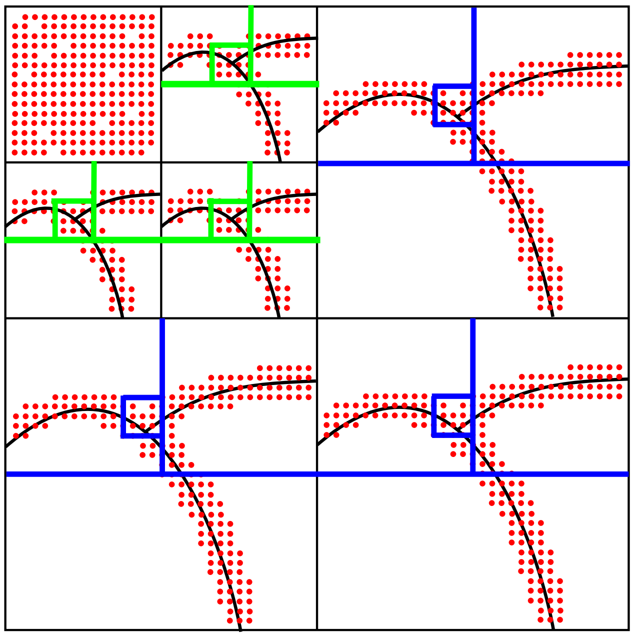

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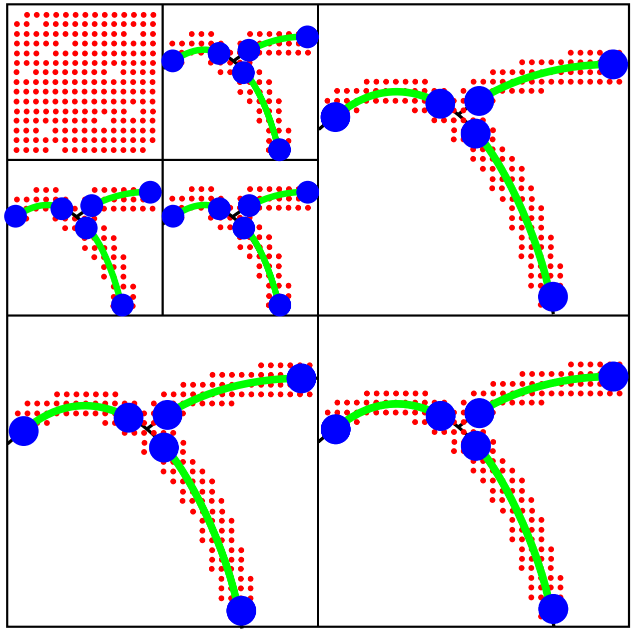

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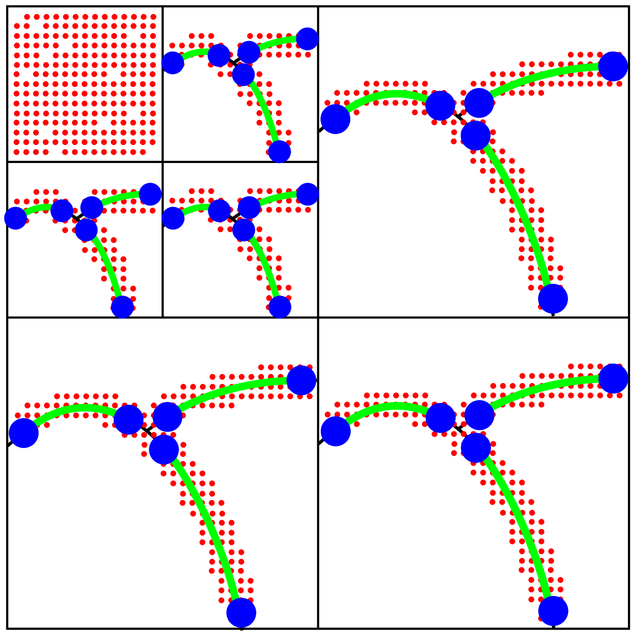

- A bandelet representation includes:
  - Beginning and ending points of bands at each scale.
  - Geometric wavelet coefficients that specify each band.
  - Bandelet coefficients in each band.
  - Wavelet coefficients outside all bands.

Bandelet Approximation Theorem

Gabriel Peyré

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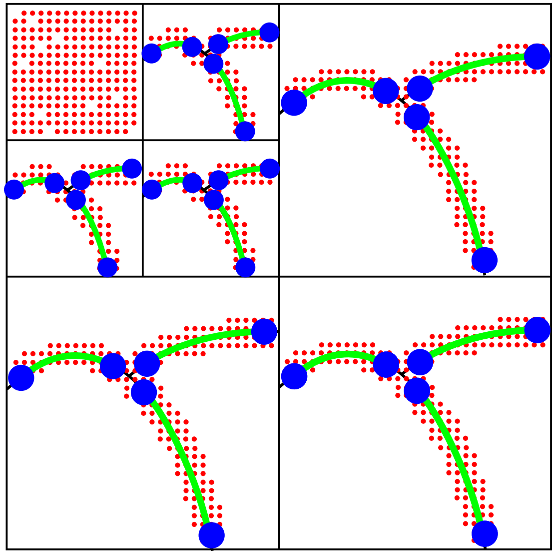
Theorem: Suppose that $\tilde{f}$ is $\mathbf{C}^\alpha$ away from "edges" that are piecewise $\mathbf{C}^\alpha$.
If $f = \tilde{f}$ or $f = \tilde{f} \star \theta_s$ then a bandelet approximation f_M, with $M = M_B + M_W + M_G$, satisfies

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \;.$$



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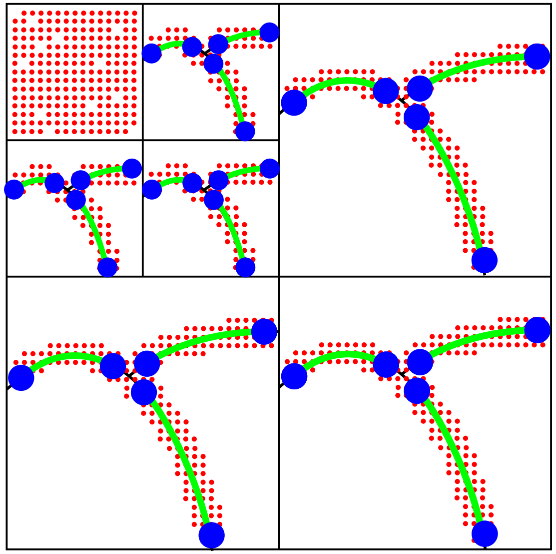

- Optimal (unknown) decay exponent α.

Numerical Experiments

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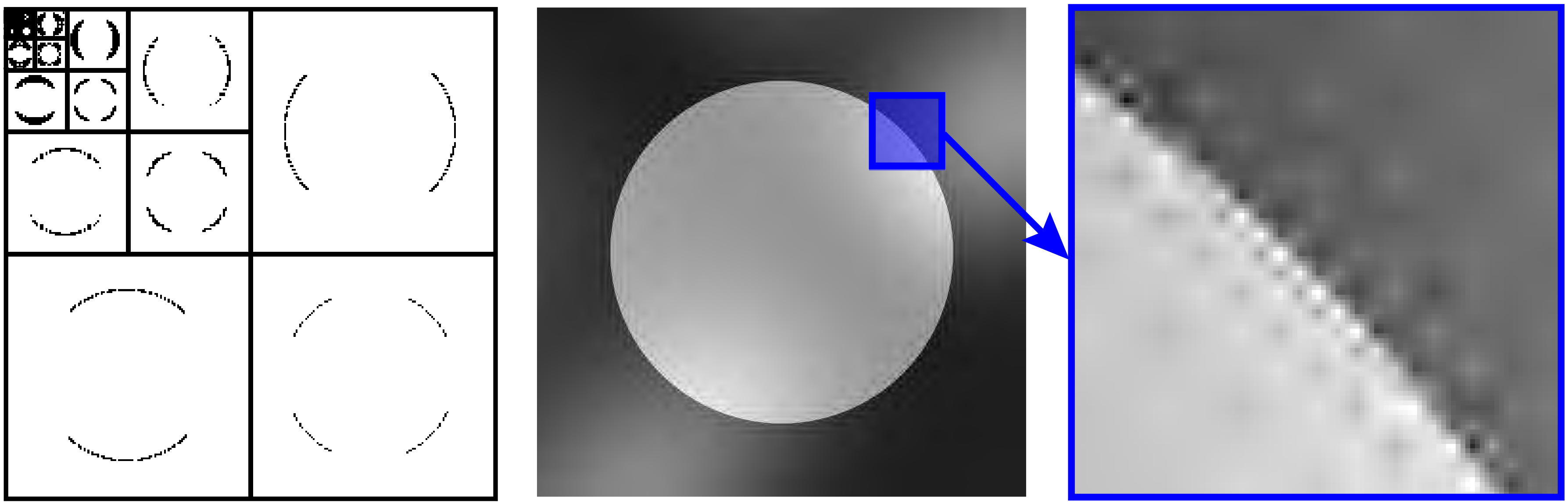
$|\langle f, \psi_{jn} \rangle| > T$

Reconstruction with
M/N^2=0.15% of coefficients



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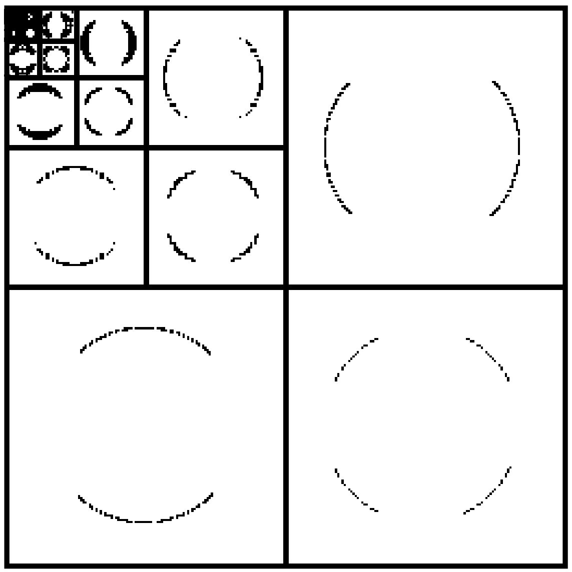

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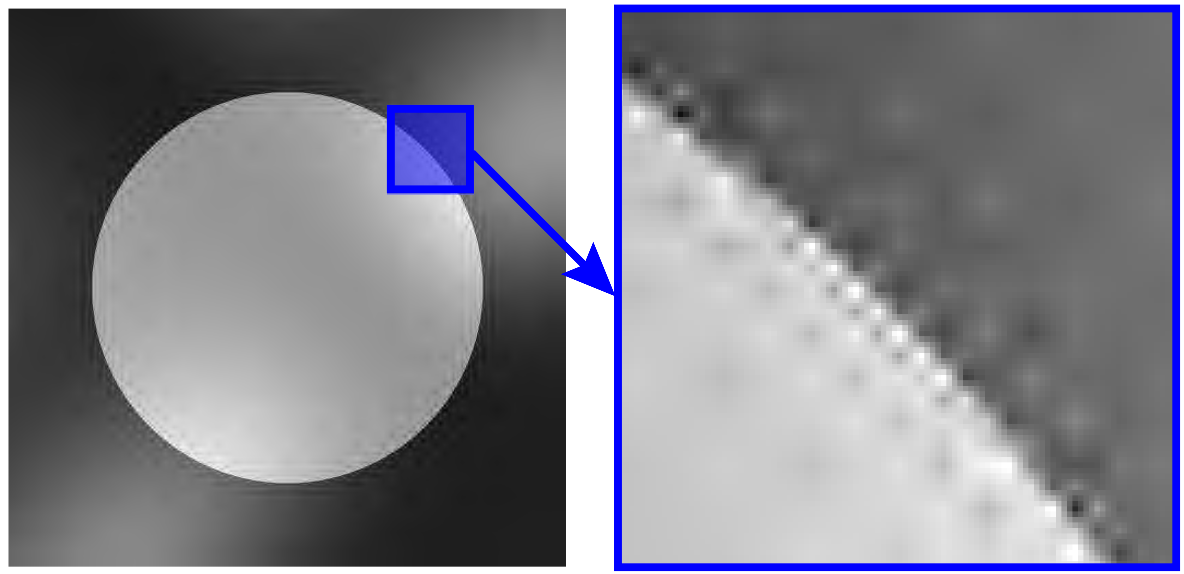

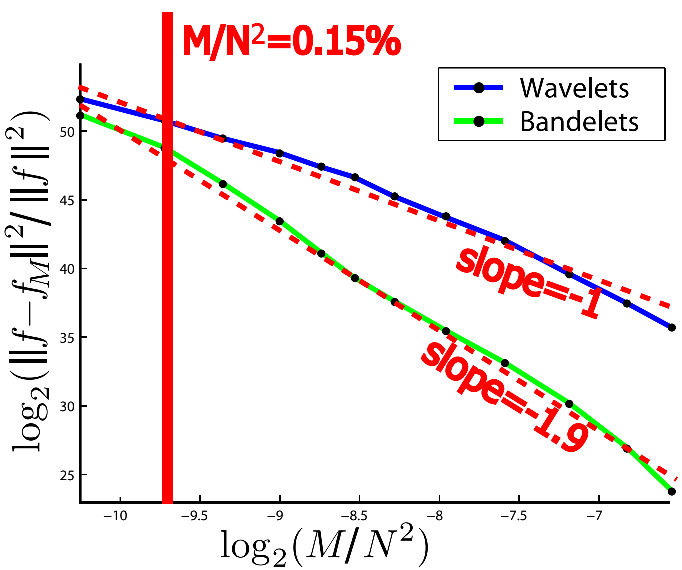

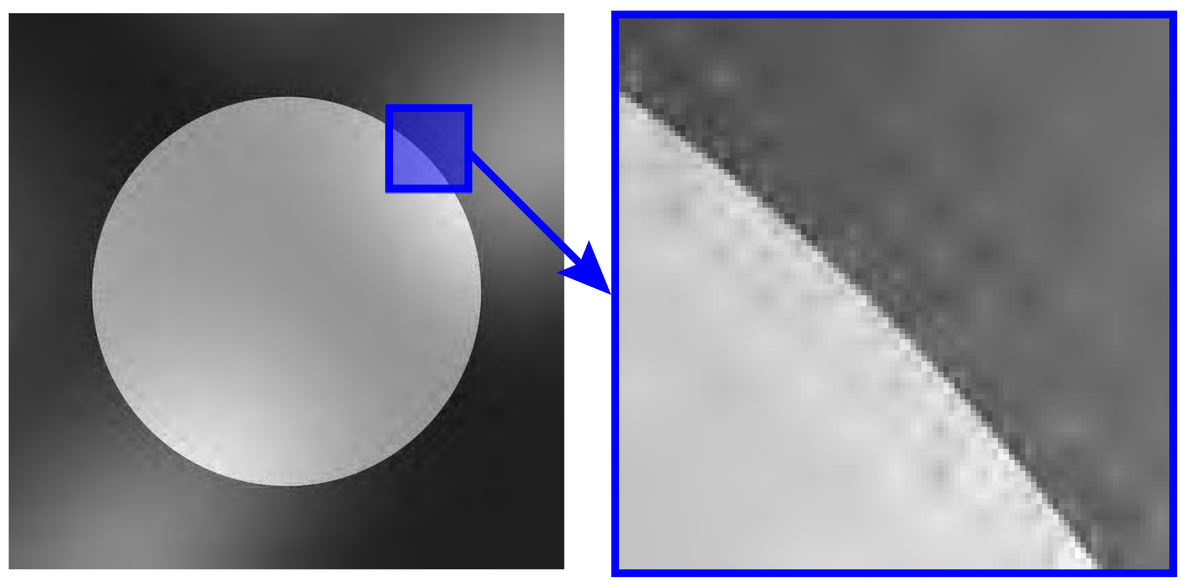

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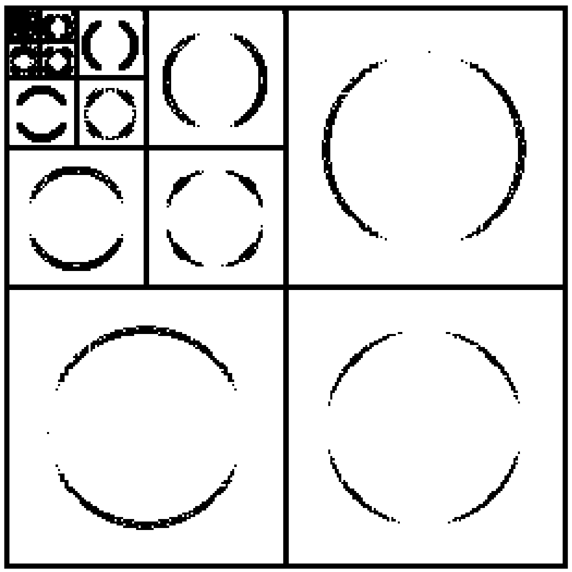

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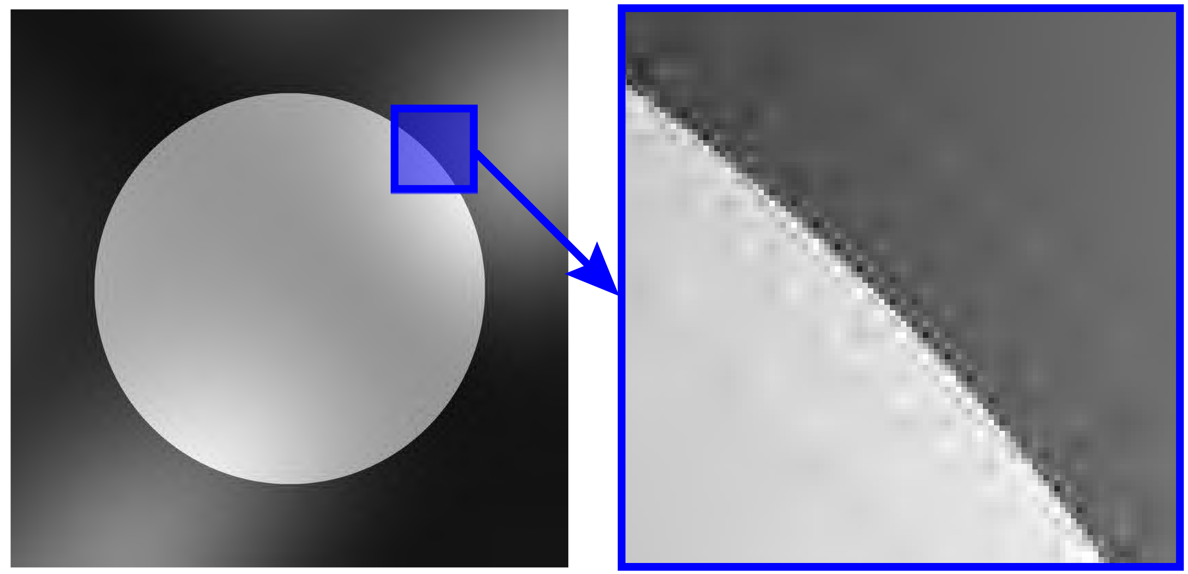

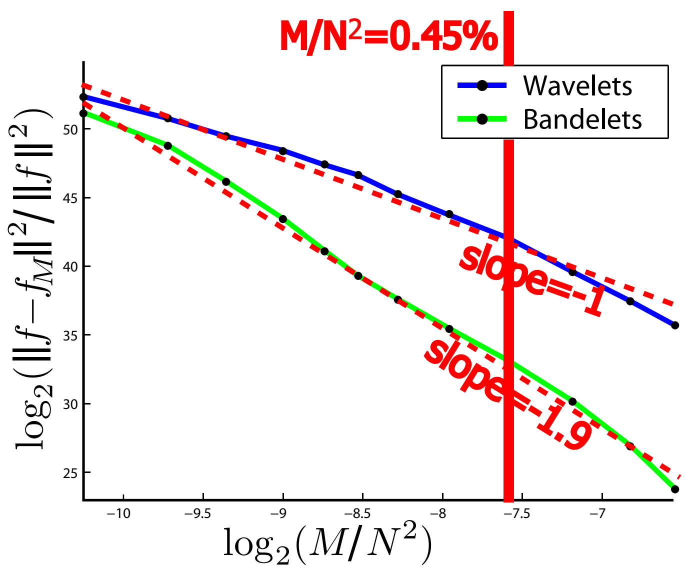

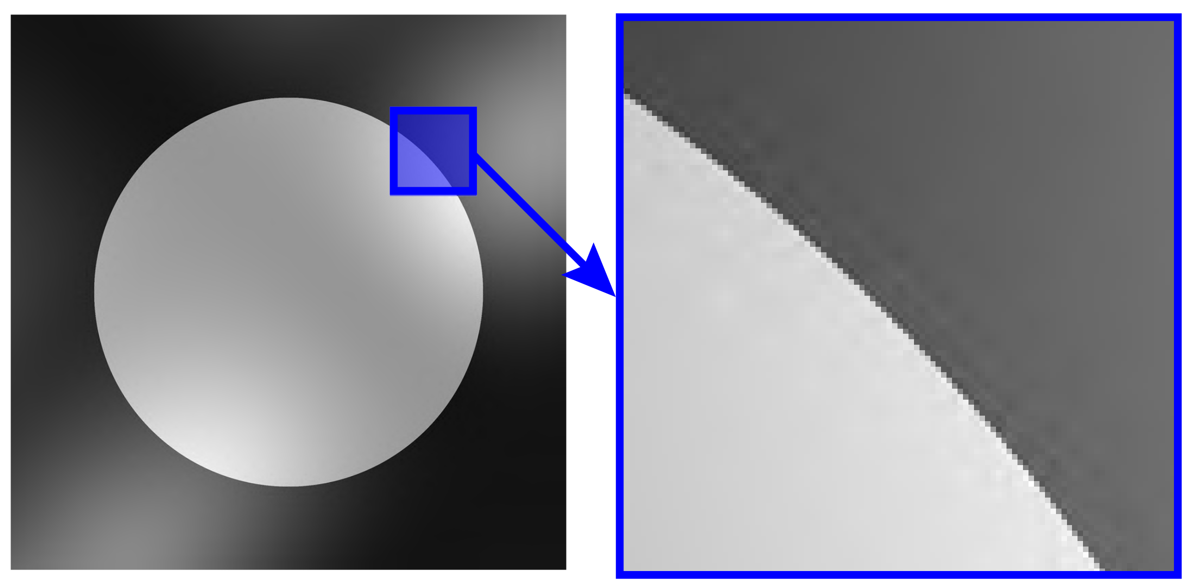

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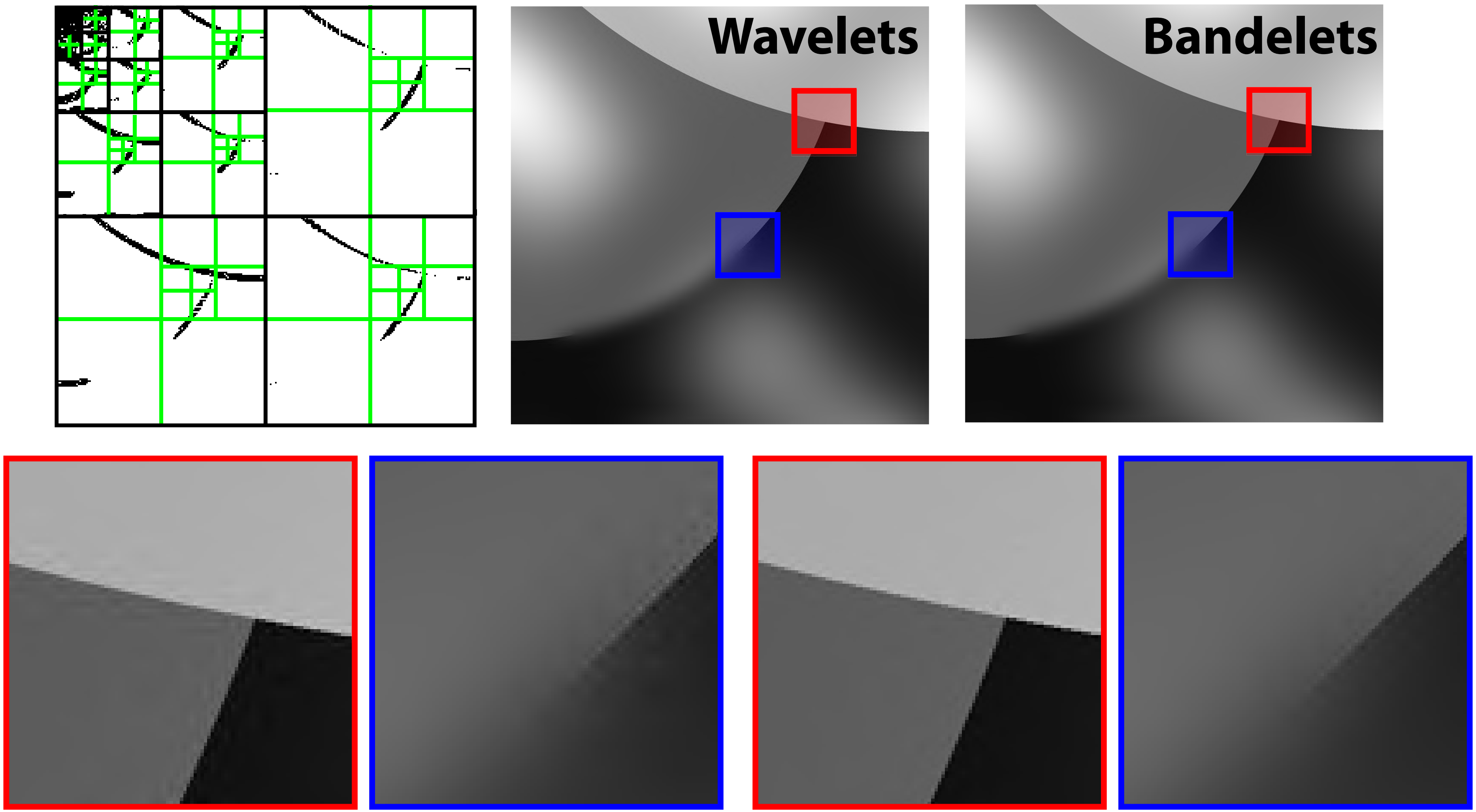

Wavelets **Bandelets**

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- Let It Wave: image compression codec adapted to the geometry of faces.

ID Photo

ID Photo

500 bytes

ID Photo

500 bytes

JPEG

ID Photo

500 bytes

JPEG

JPEG-2000

ID Photo

500 bytes

JPEG

JPEG-2000

Bandelets
LET IT WAVE

ID Photos

ID Photos

JPEG JPEG-2000 LIW

500 bytes

ID Photos

JPEG | JPEG-2000 | LIW

800 bytes

ID Photos

JPEG | JPEG-2000 | LIW

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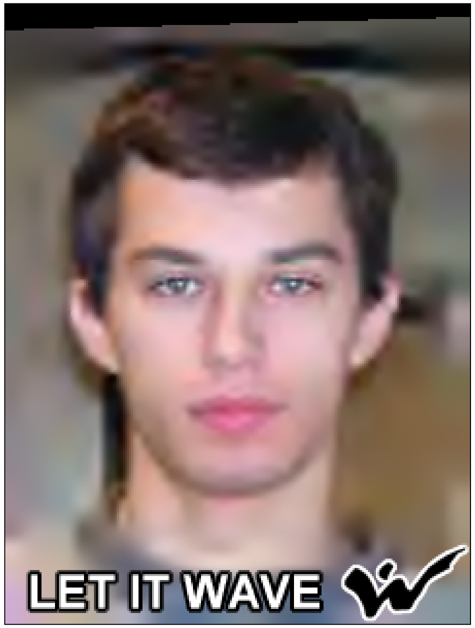
ID Photos



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