Geometric Image Representation with Bandelets

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- Need to take advantage of geometrical image regularity to improve representations.
- Building harmonic analysis representations adapted to complex $geometry$.

Edge Detection

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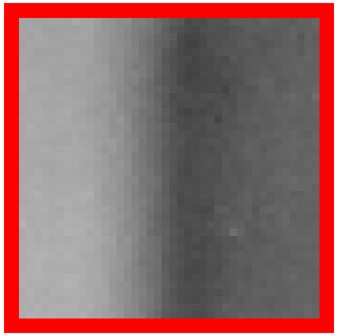

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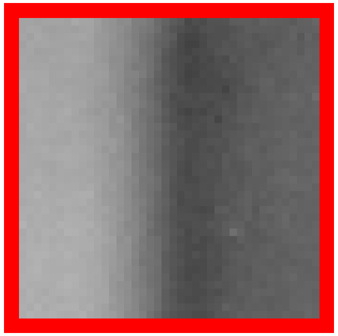

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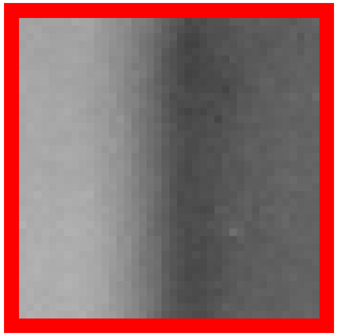

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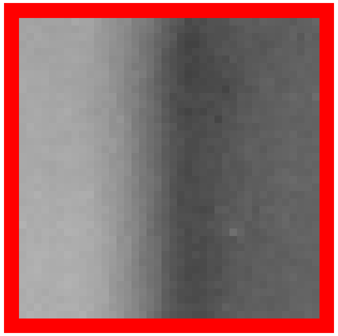

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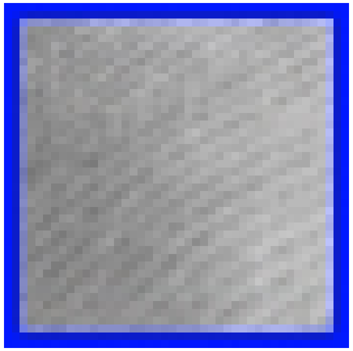

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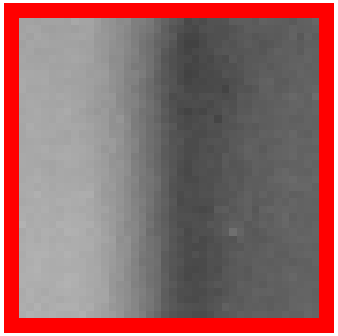

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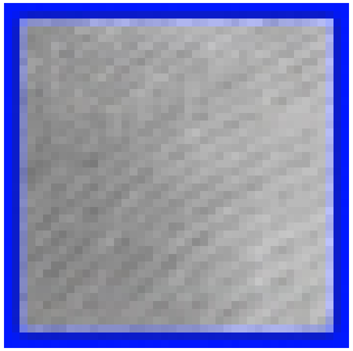

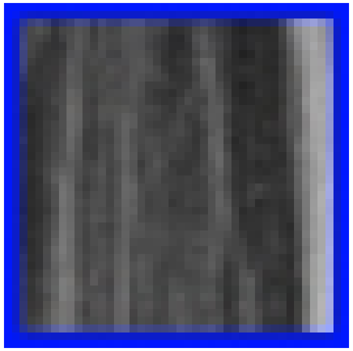

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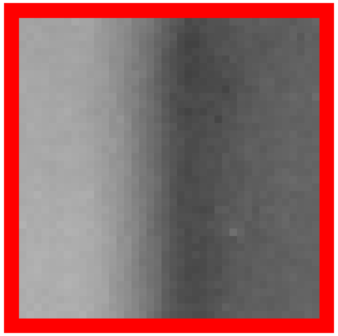


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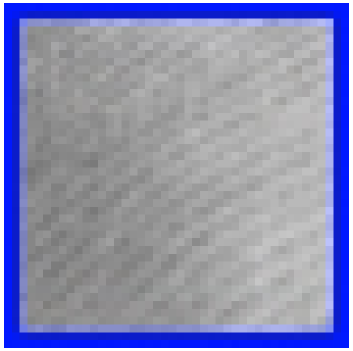
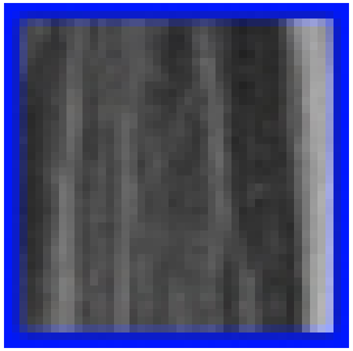
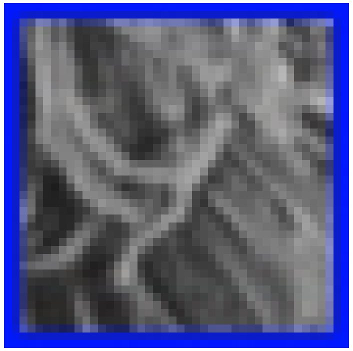

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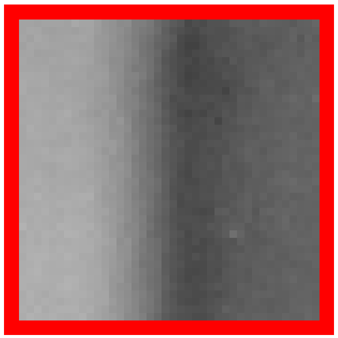


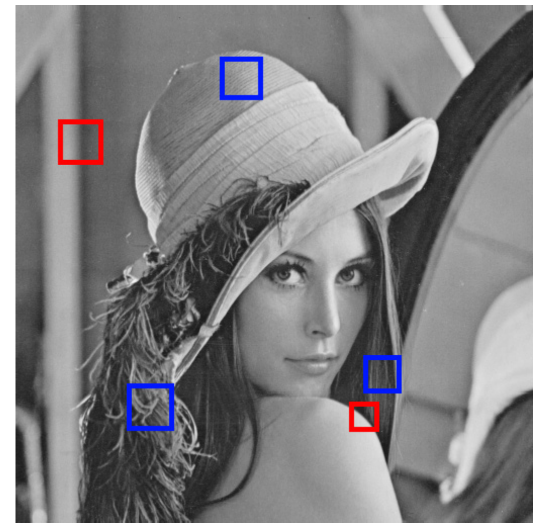

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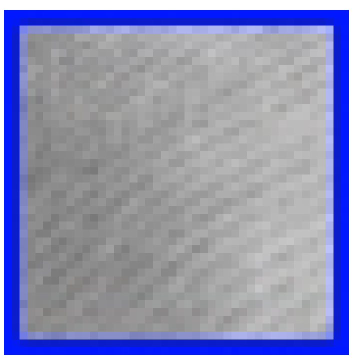
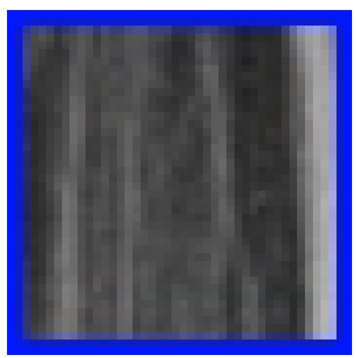
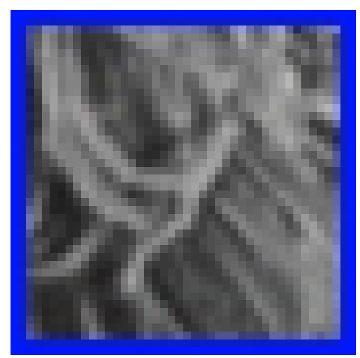

- How can the estimation of the geometry become well-posed ?

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- Bandelets and non linear approximation.
- Application to compression and denoising.

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- **Problem:** Given that $f \in \Theta$, how to choose $\mathbf{B}$ so that

$$\|f - f_M\|^2 \leqslant C M^{-\beta} \ \text{ with } \beta \text{ large ?}$$

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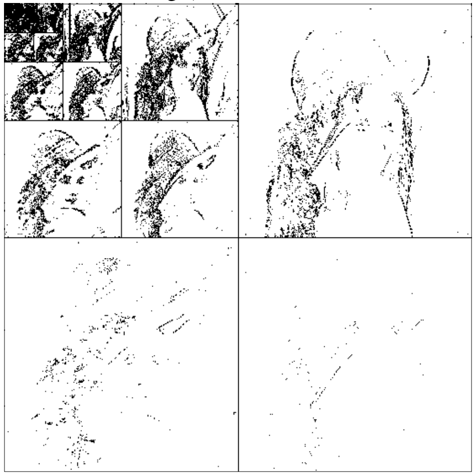

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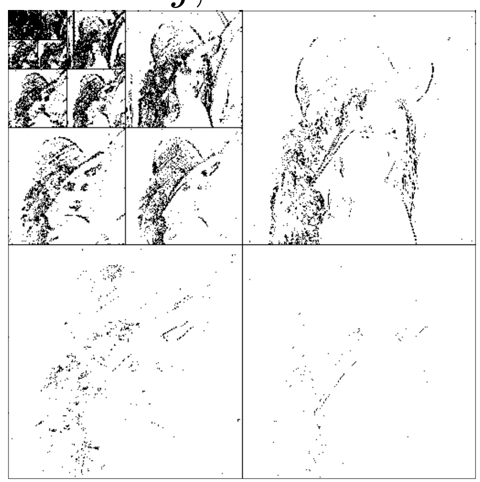

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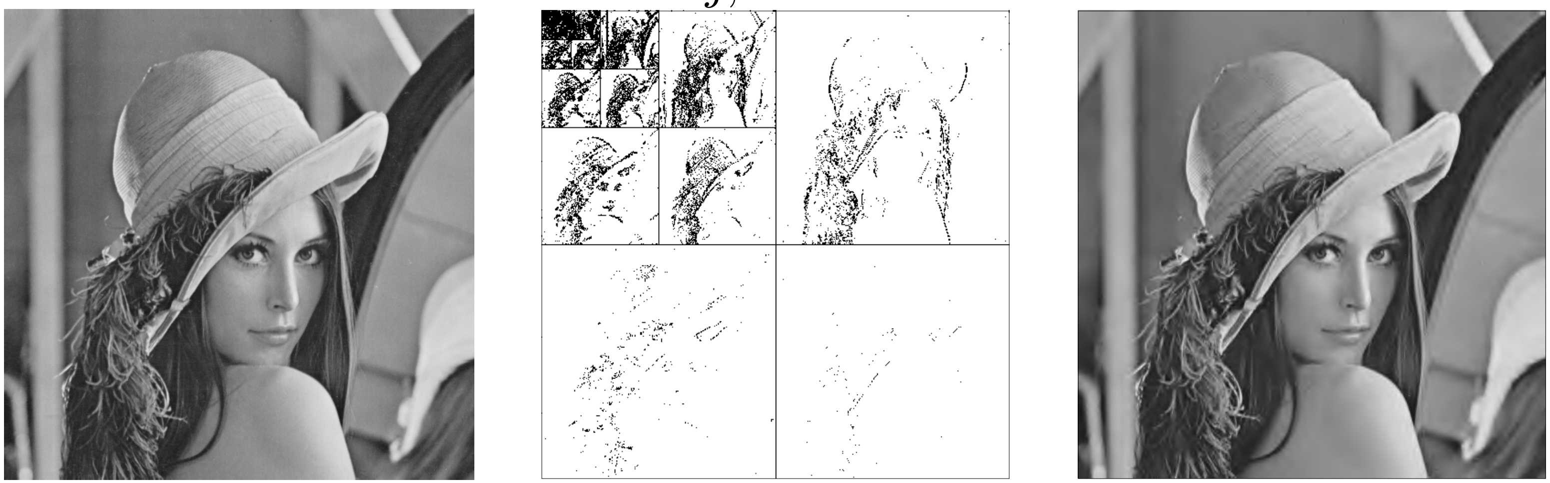

- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.
- But: does not take advantage of any geometric regularity.

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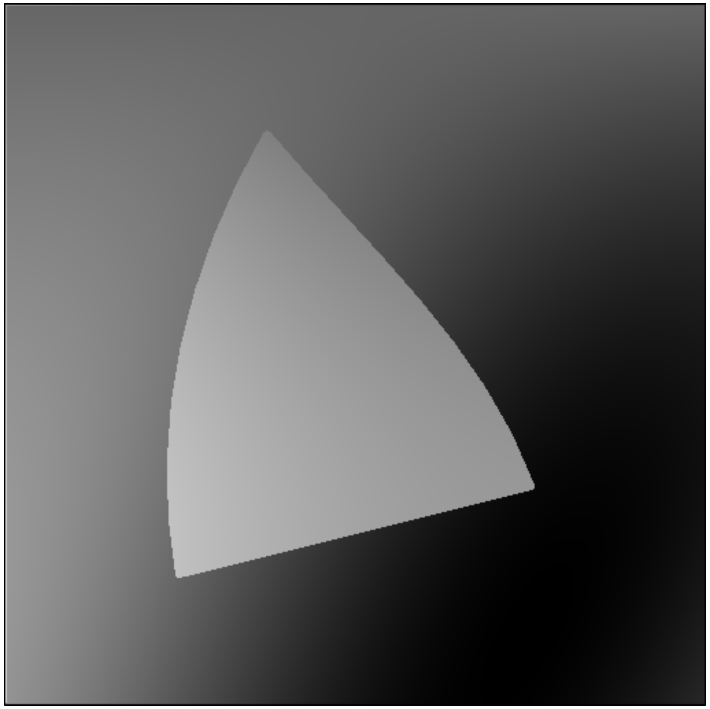

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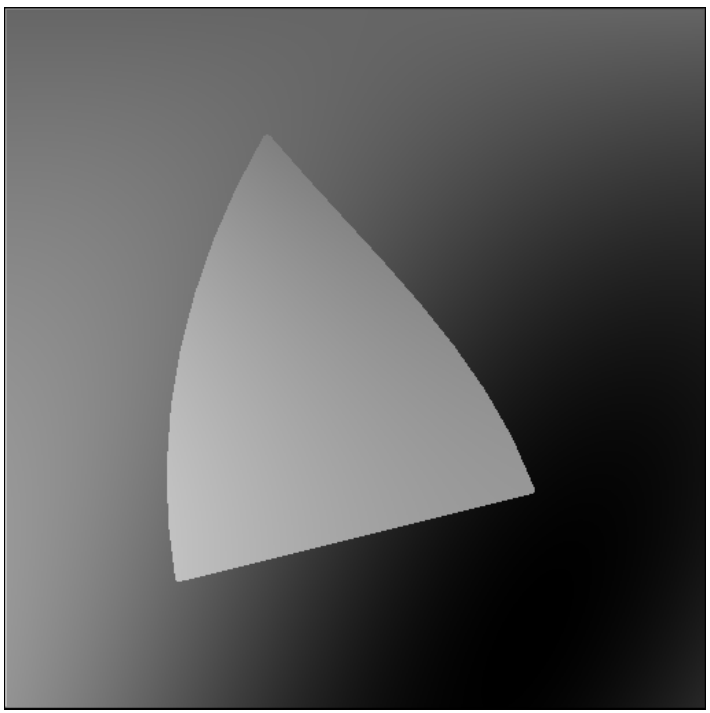

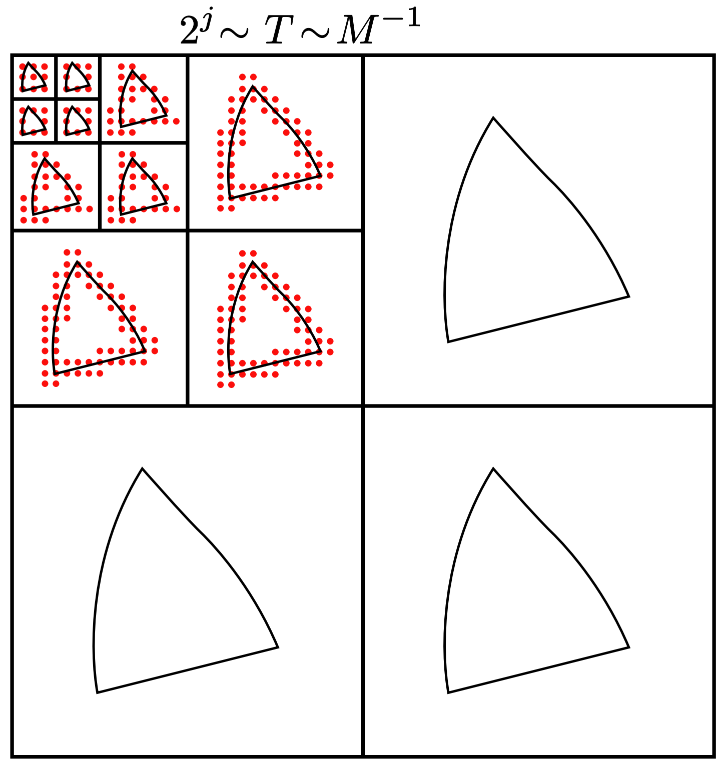

- with M wavelets: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.

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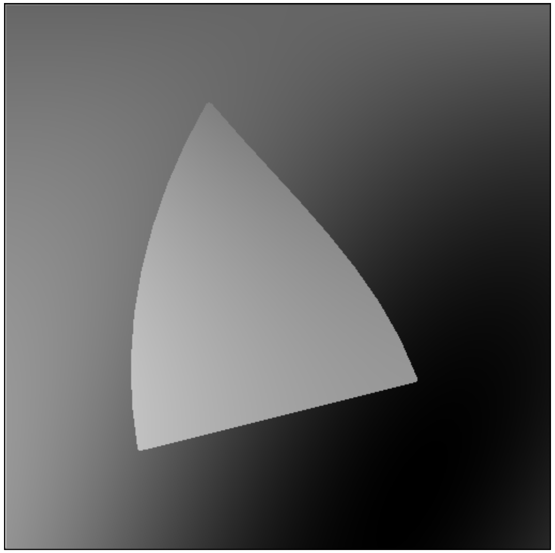

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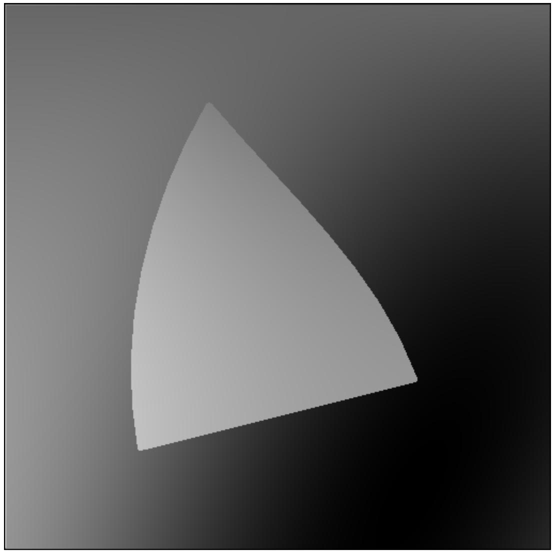
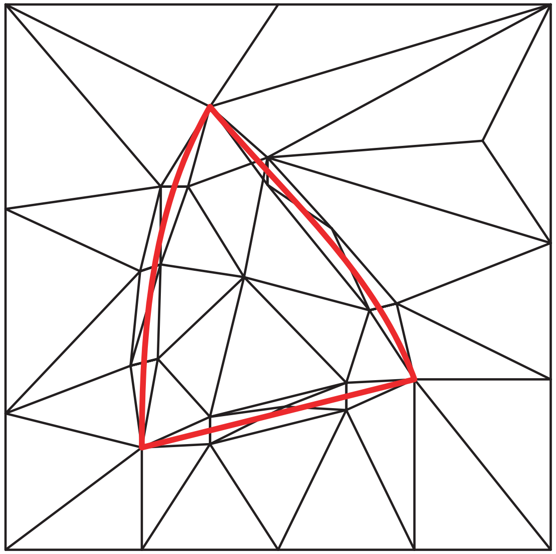
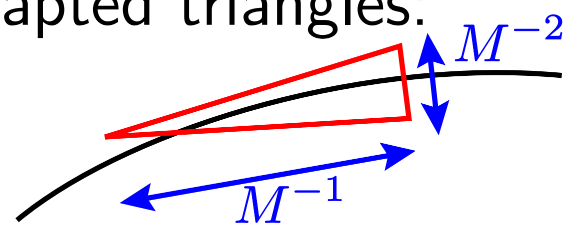




- Piecewise linear approximation over M adapted triangles: if $\alpha \geqslant 2$ then $\|f - f_M\|^2 \leqslant C\, M^{-2}$,



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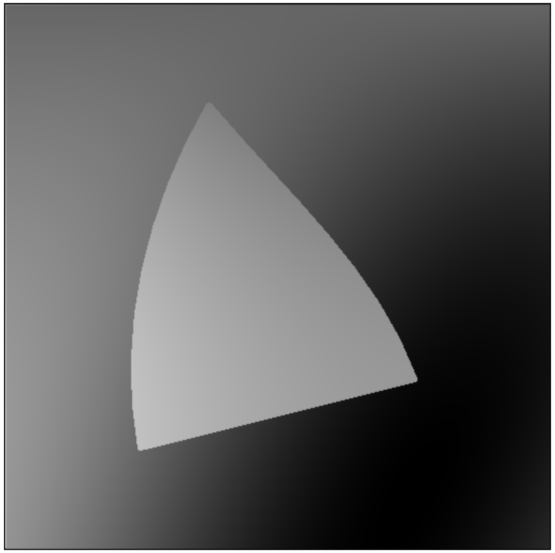

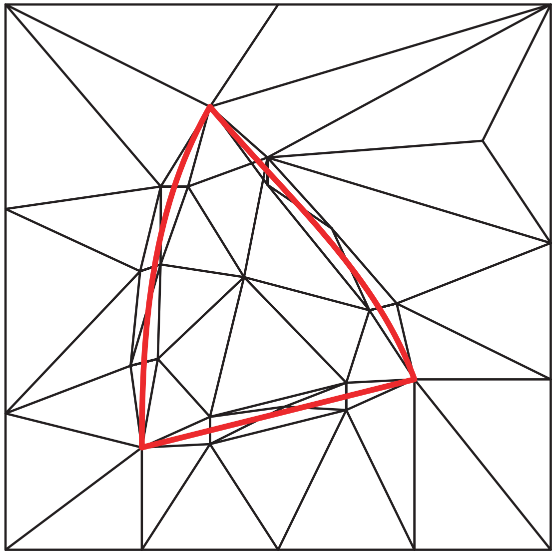

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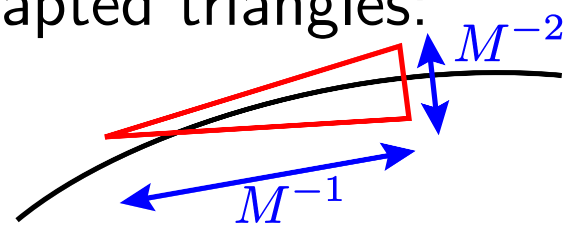
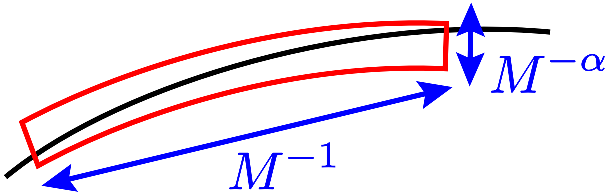


- Higher order approximation over M adapted "elements": $\|f - f_M\|^2 \leqslant C\, M^{-\alpha}$.



Adaptive Triangulation for Smooth Edges

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- Approximations of $f = \tilde{f} \star h_s$ which:
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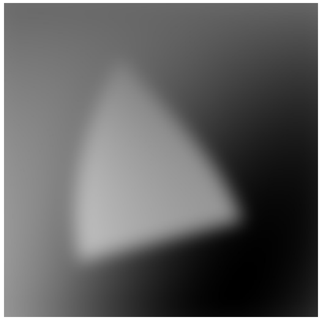

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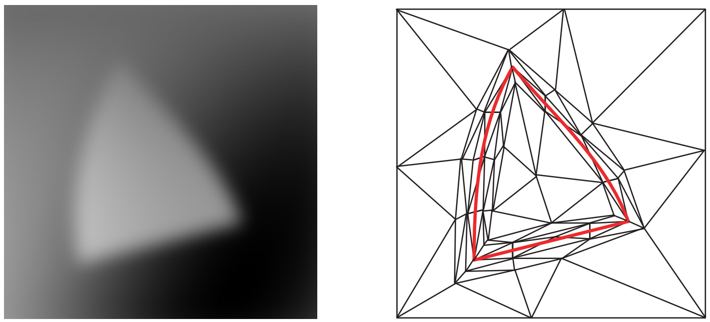

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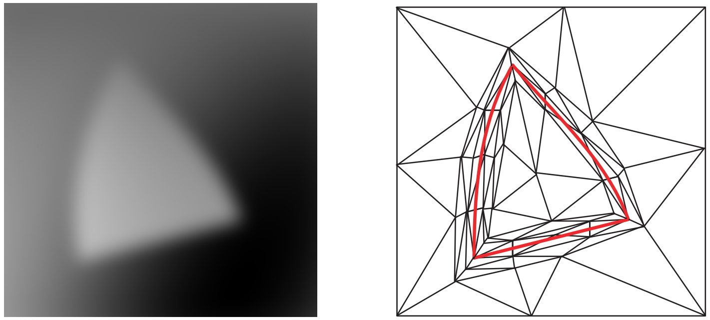
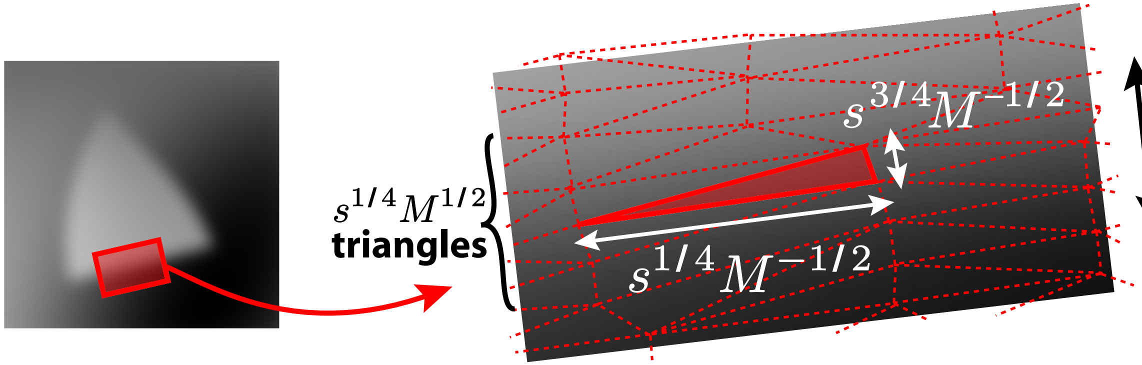


- With M adaptive triangles: $\|f - f_M\|^2 \leqslant C\, M^{-2}$.



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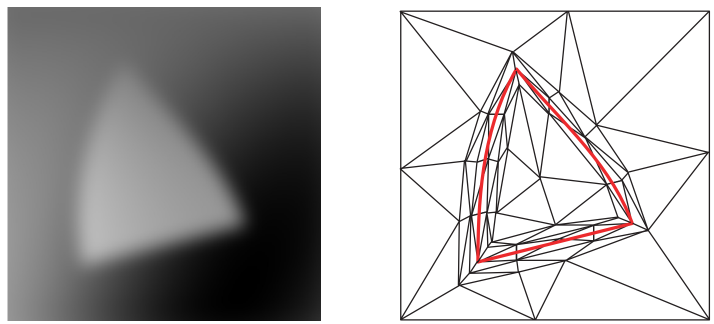

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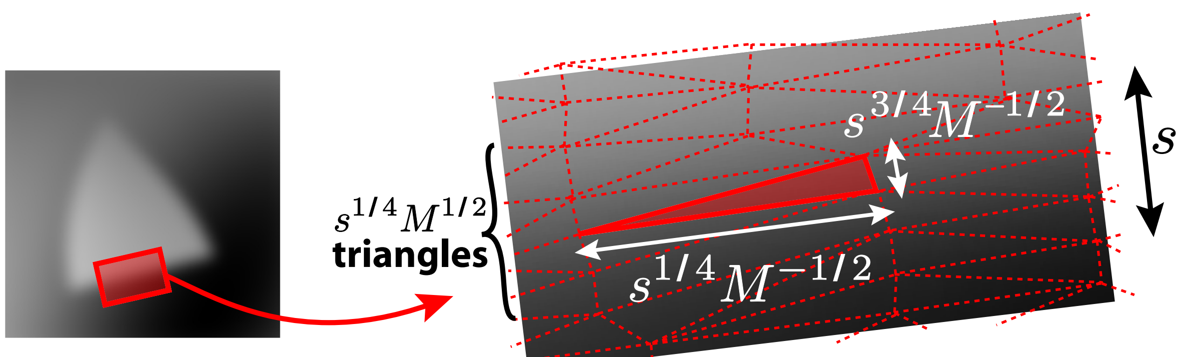

- Difficult to find optimal approximations but good greedy solutions (*Dekel,Demaret, Dyn, Iske*).

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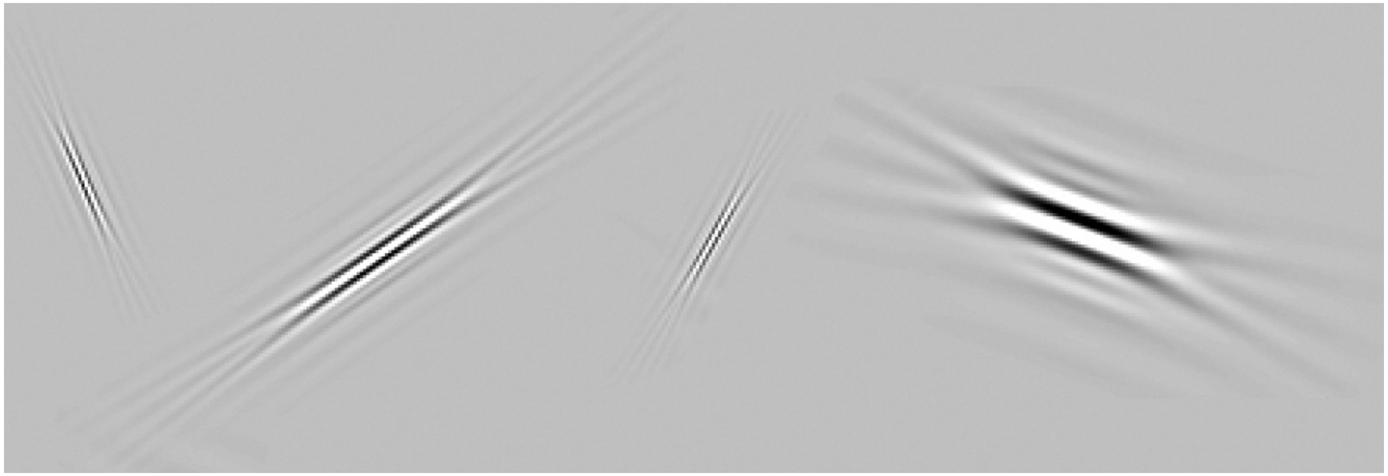
- Curvelets define tight frames of $L^2[0,1]^2$ with elongated and rotated elements (*Candes, Donoho*): $\{c_j(R_\theta x - \eta)\}_{j,\theta,\eta}$



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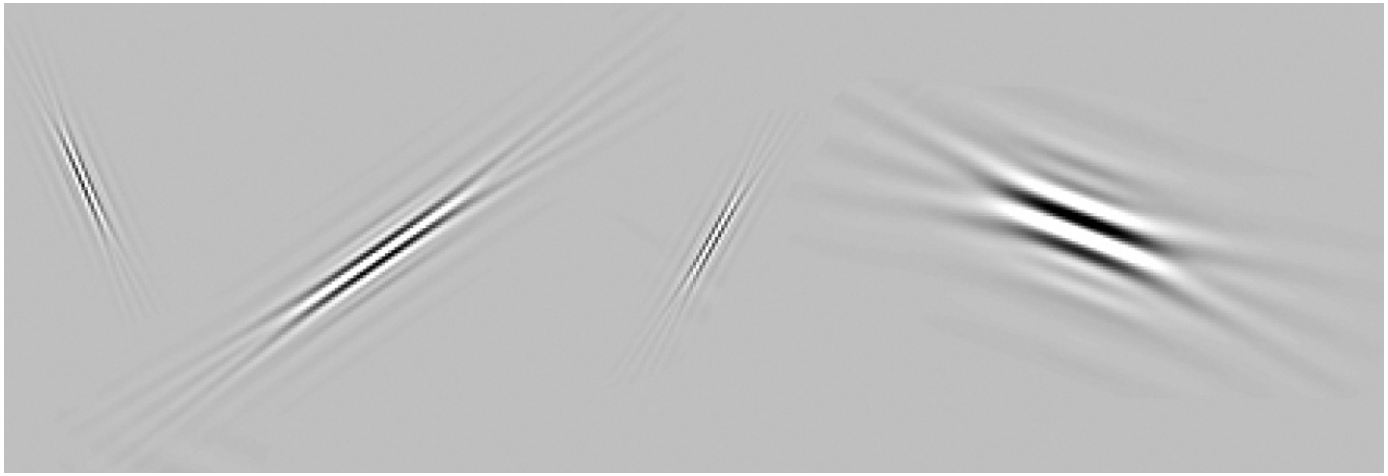

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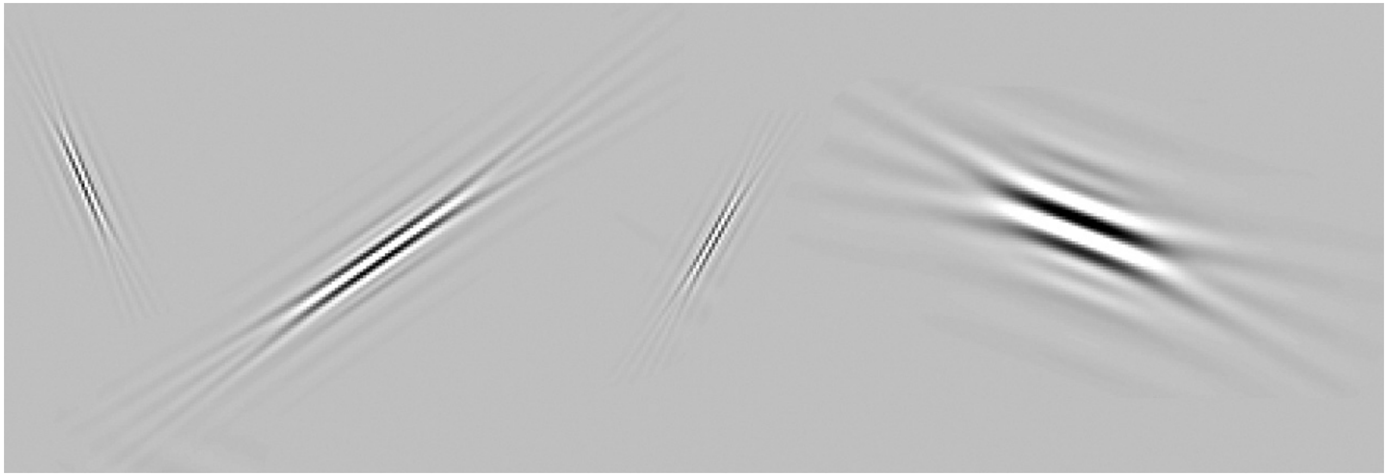

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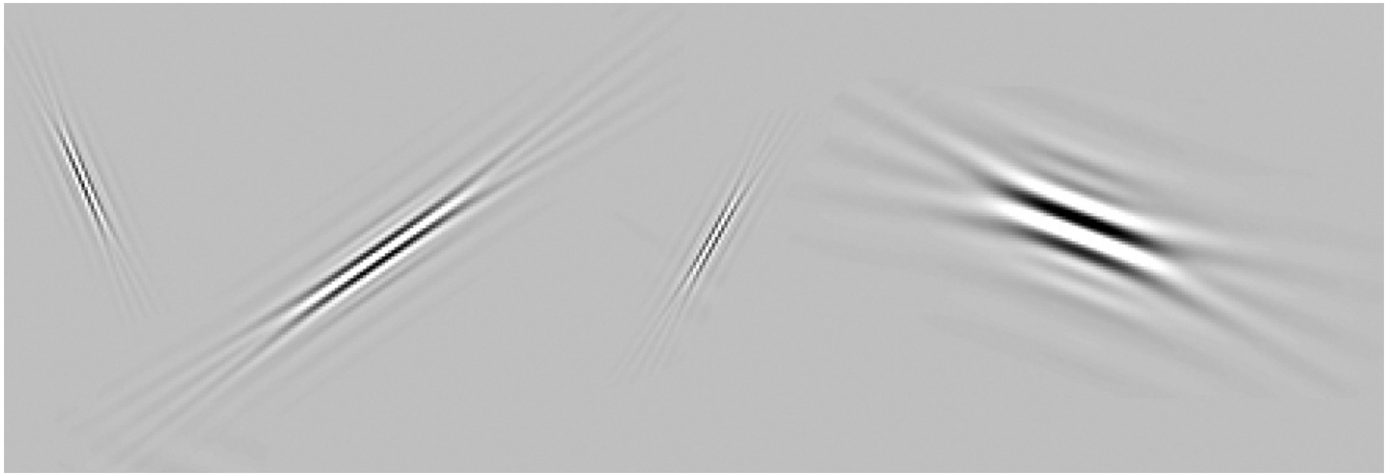

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- Difficulty to build discrete orthogonal/Riesz bases: (*Vetterli & Minh Do*).

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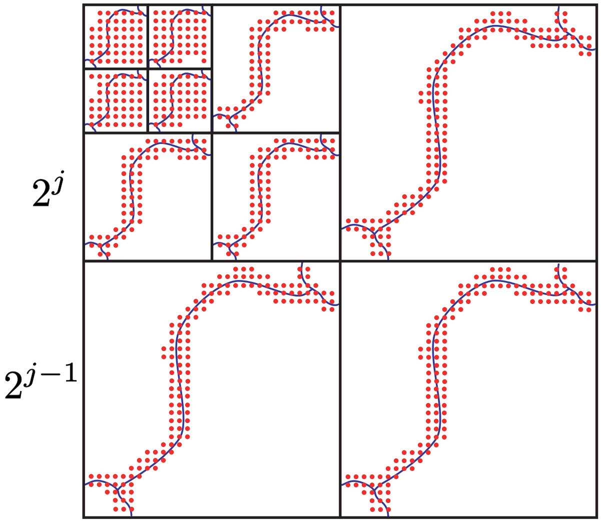

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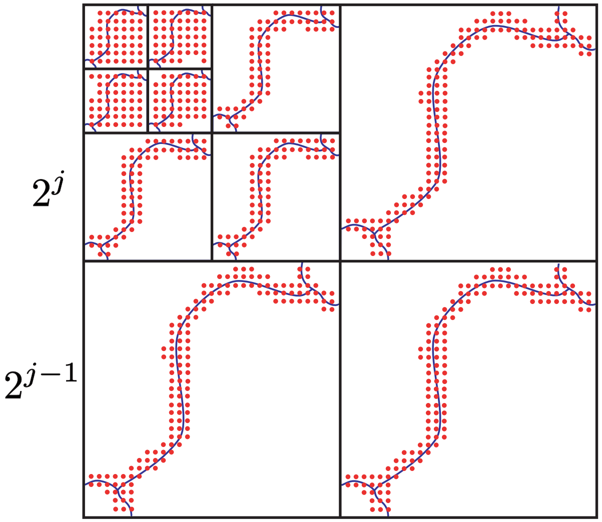

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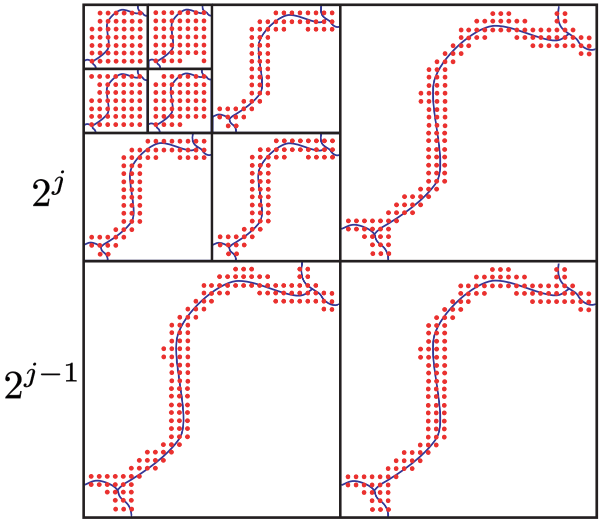

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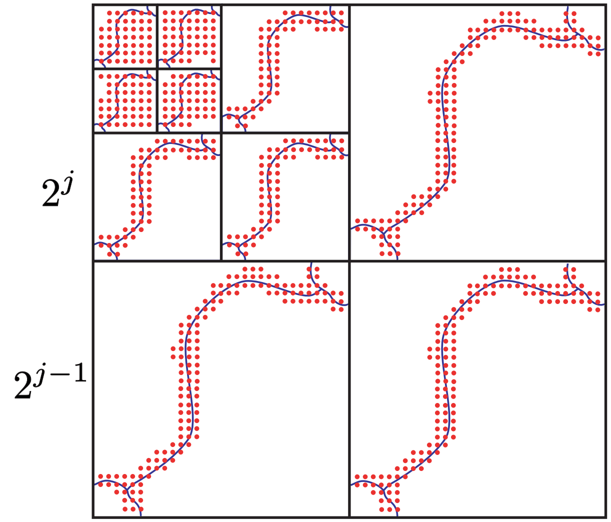

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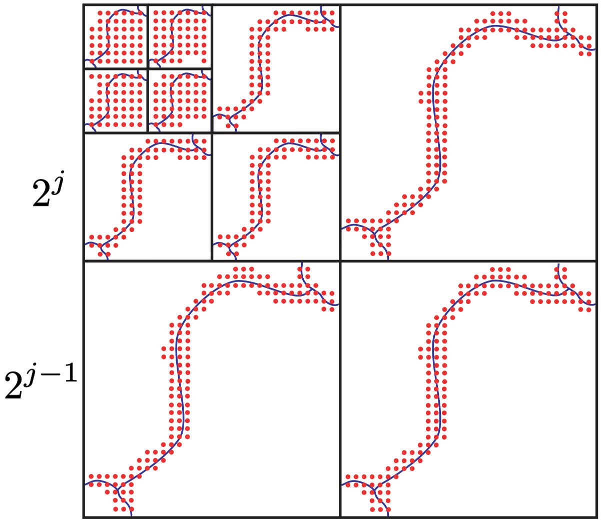

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- Modification of the wavelet transform (*Cohen*).

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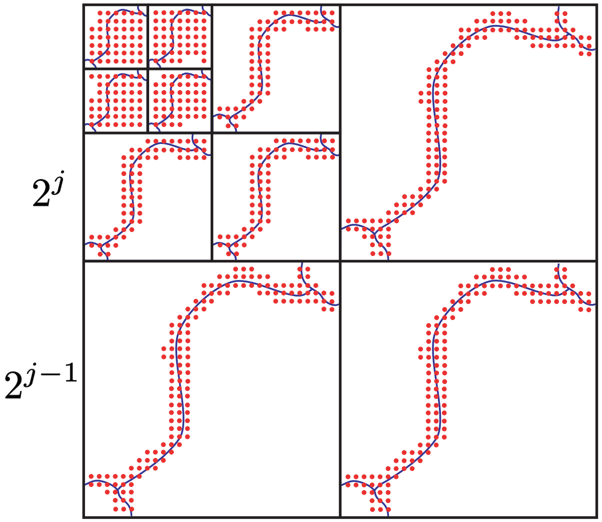

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- Modification of the wavelet transform (*Cohen*).
- Bandelets NG (*Peyré*) (more information in Zürich).

Geometric Model 1

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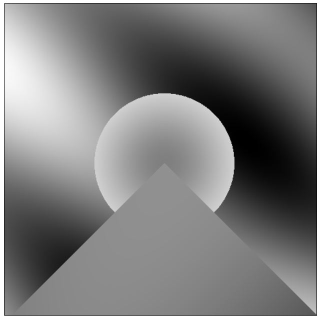
- By parts regular functions with discontinuities along regular curves:



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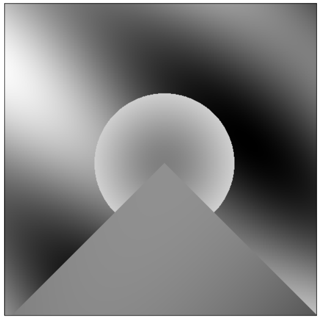
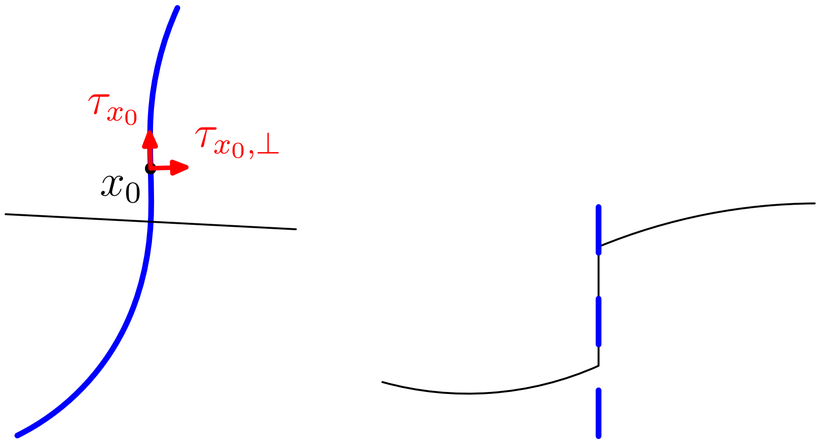


- True discontinuities:



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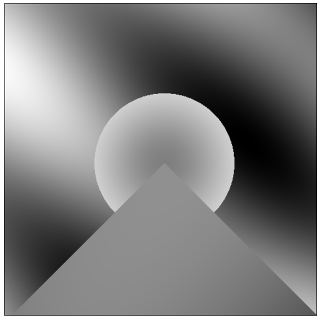

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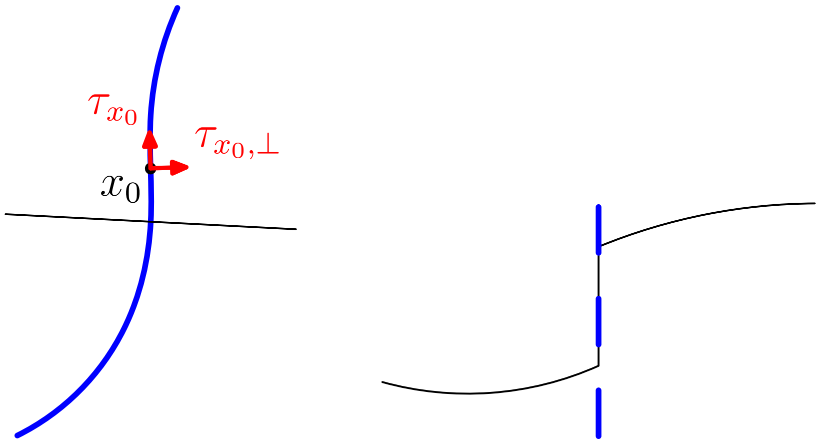
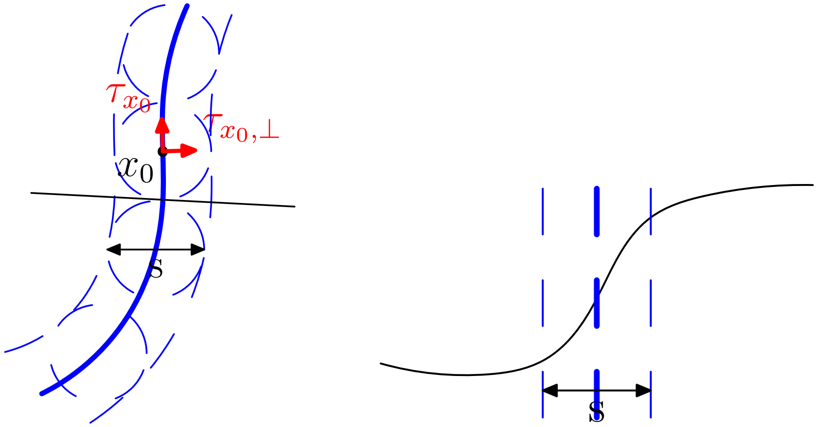


- Smoothed discontinuities:



Geometric Model 2

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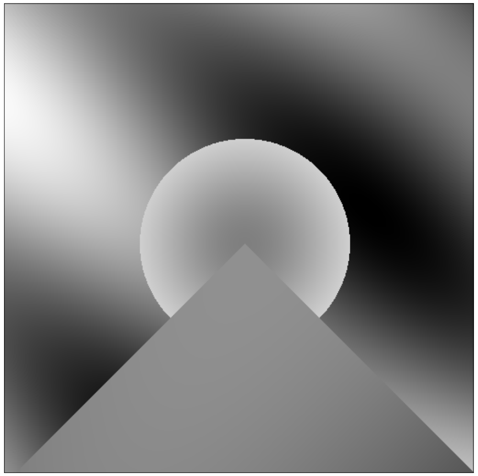

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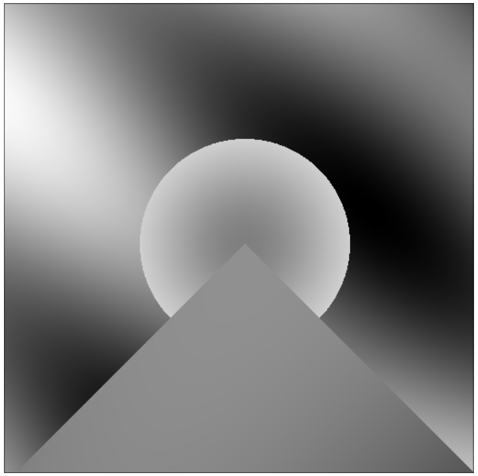

- $\mathbf{C}^{\alpha}$ Horizon Model of Donoho revisited.

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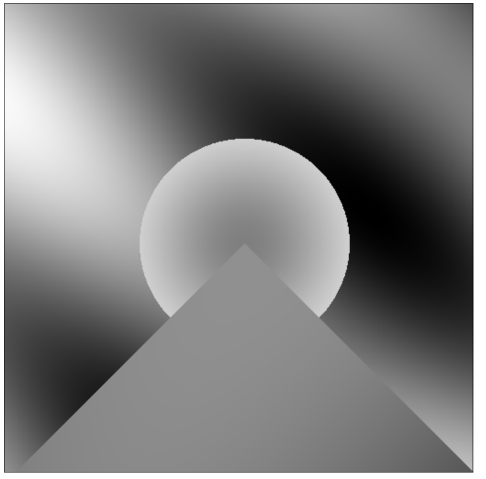

- $\mathbf{C}^\alpha$ Horizon Model of Donoho revisited.
- $\mathbf{C}^\alpha$ Geometrically Regular:
  - $f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f} \in \mathbf{C}^\alpha(\Lambda)$ for $\Lambda = [0,1]^2 - \{\mathcal{C}_\gamma\}_{1 \leqslant \gamma \leqslant G}$,
  - the blurring kernel h is $\mathbf{C}^\alpha$, compactly supported in $[-s, s]^2$ and $\|h\|_{\mathbf{C}^\alpha} \leqslant s^{-(2+\alpha)}$.
  - the edge curves $\mathcal{C}_\gamma$ are α differentiable and do not intersect tangentially.

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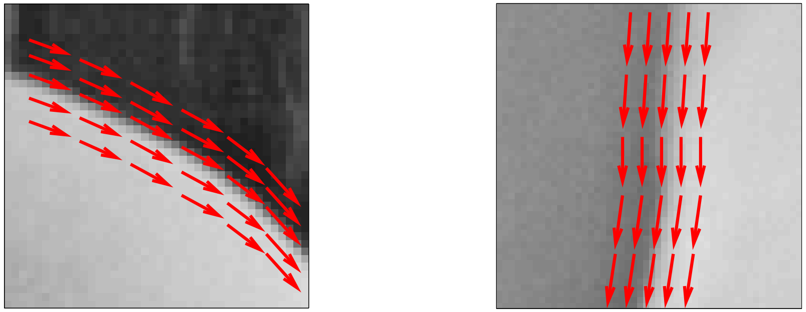

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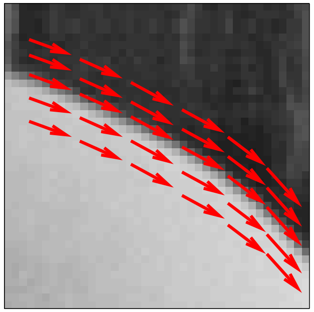
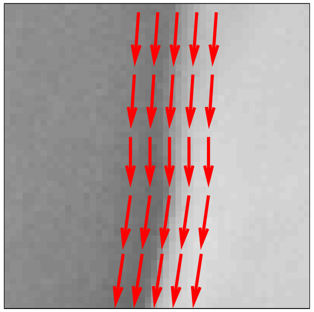
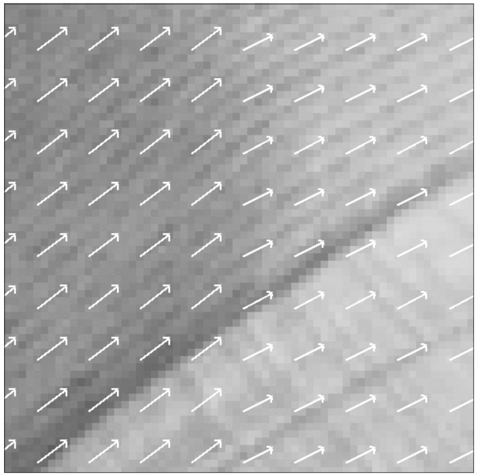
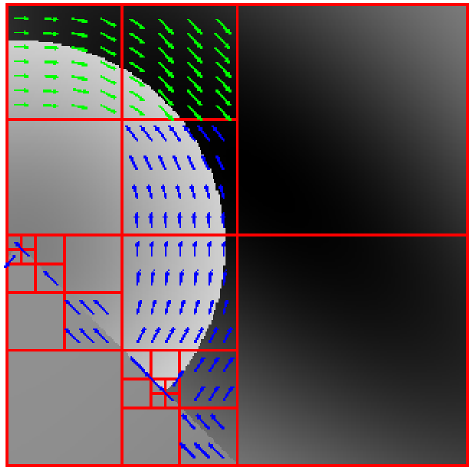
 

- The image is segmented in such regions:

Warped Wavelet Basis

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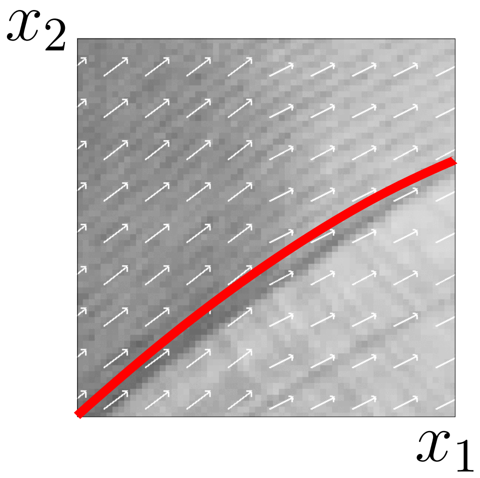
$$\vec{\tau}(x_1, x_2) = (1\,,\; c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

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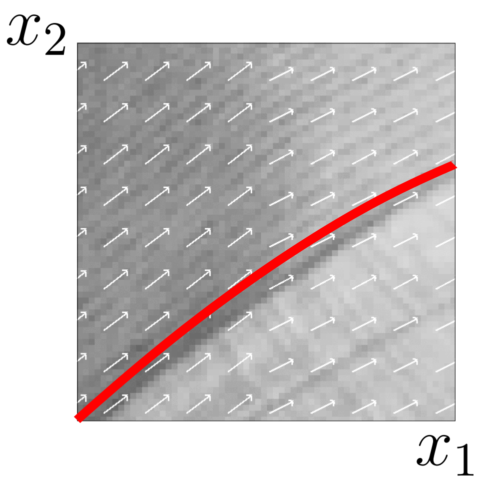

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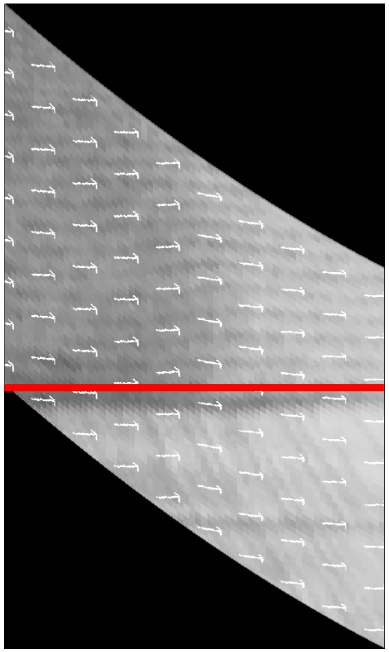

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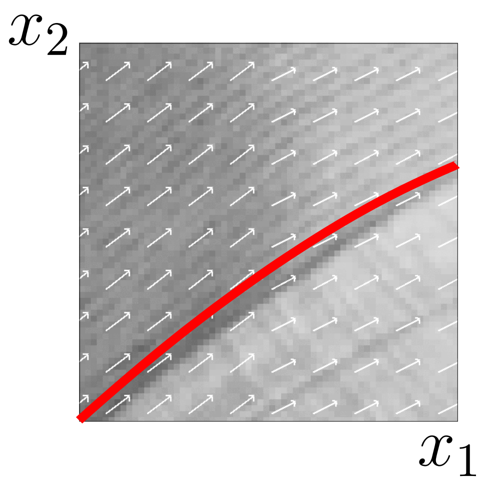

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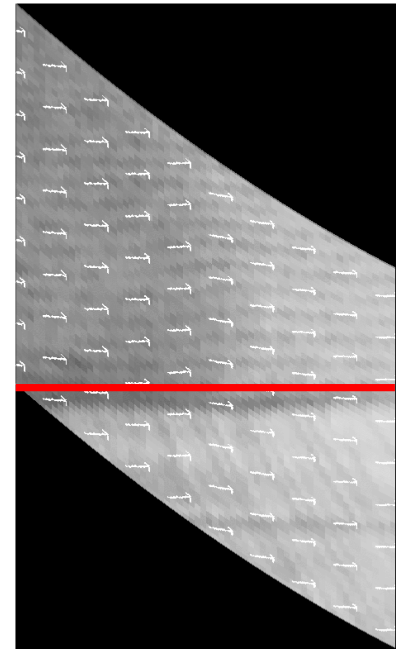

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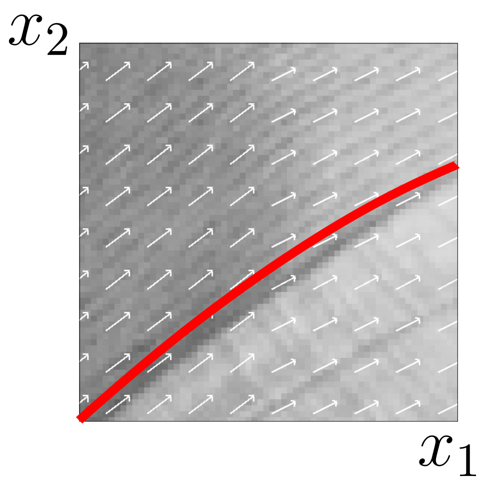

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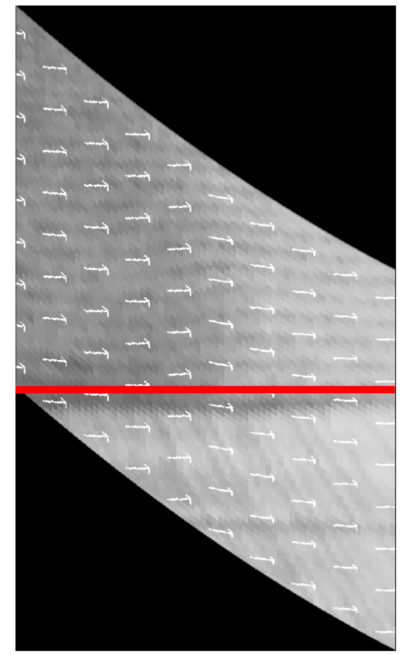

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- Decomposition in a *warped wavelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}.$$

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Anisotropic

d

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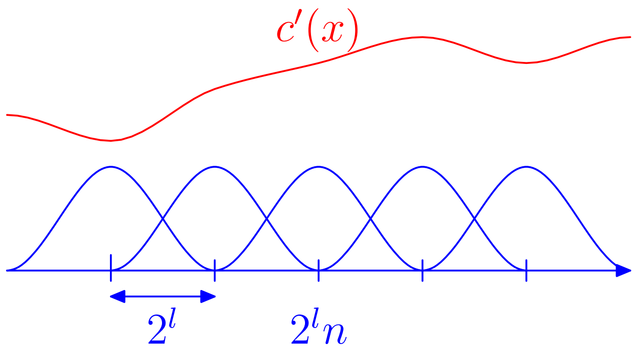

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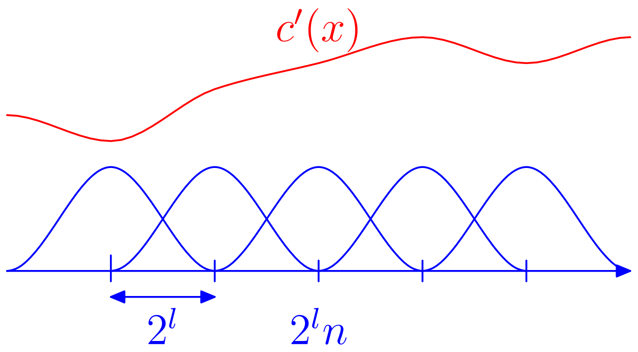
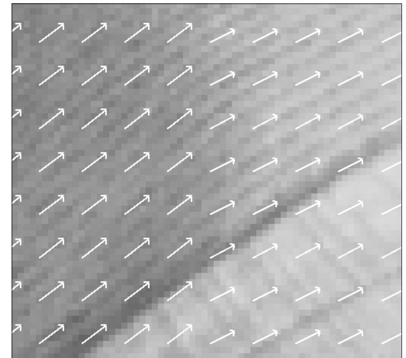


and the $L\,2^{-l}$ parameters α_n.

- Minimization of

$$\int_\Omega \left| \nabla f(x_1, x_2) \, . \, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \, .$$



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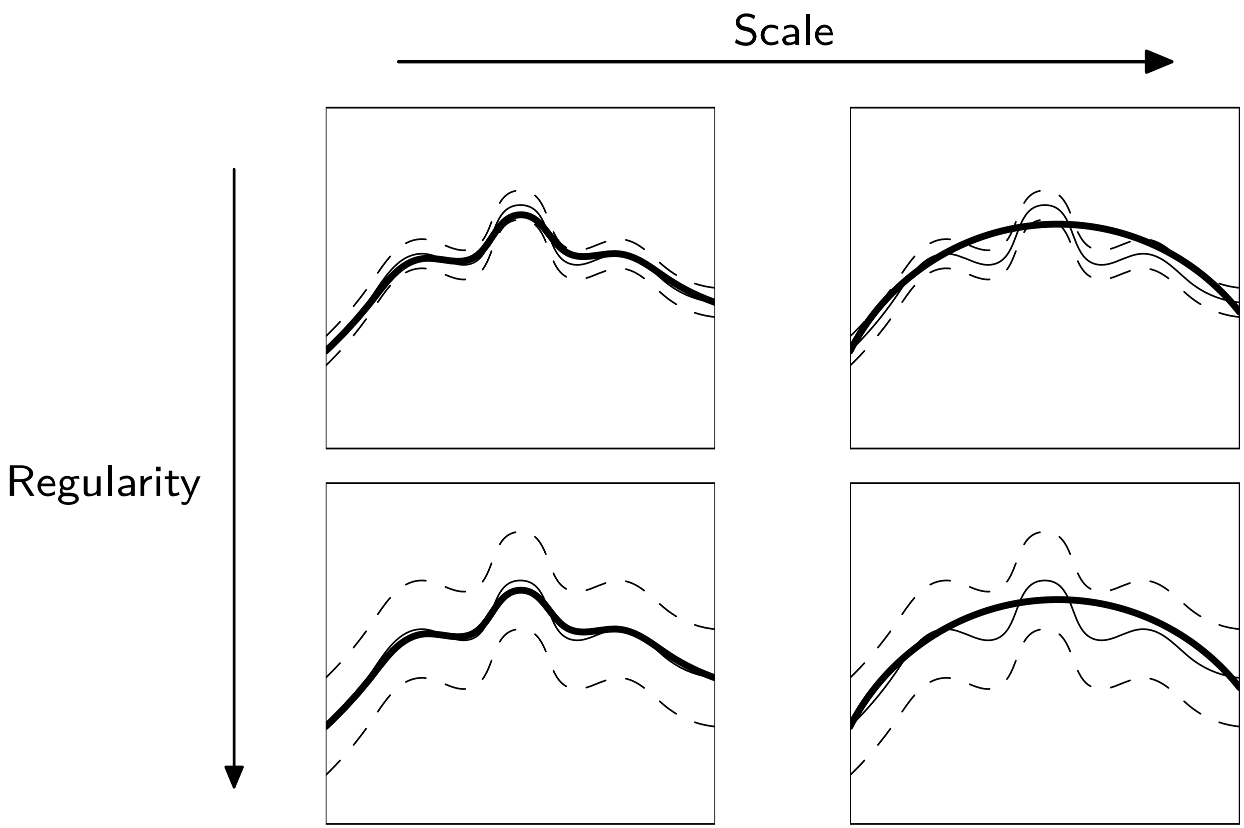

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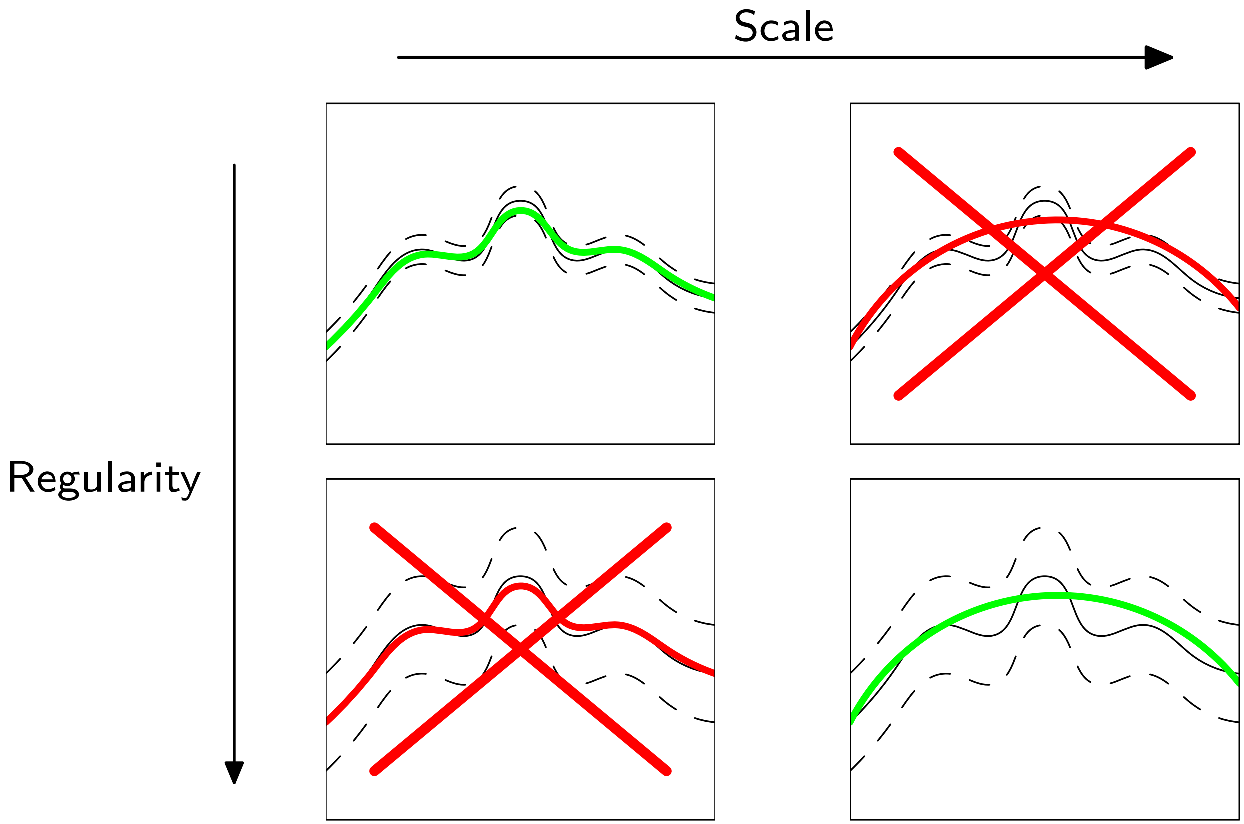

M Term Approximation

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- A bandelet approximation is specified by:
  - a dyadic squares segmentation given by the M_s interior nodes of a quadtree,
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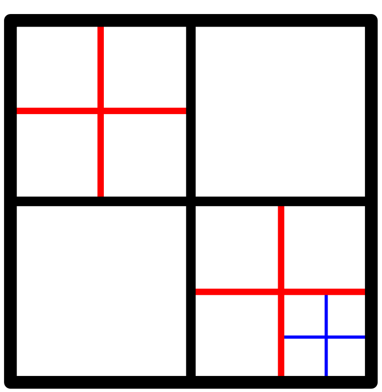
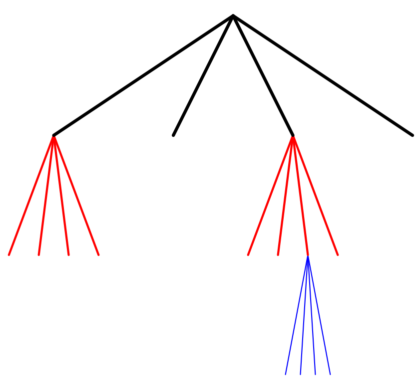
- Total number of parameters:

$$M = M_s + \sum_i \left(M_{g,i} + M_{b,i} \right) .$$

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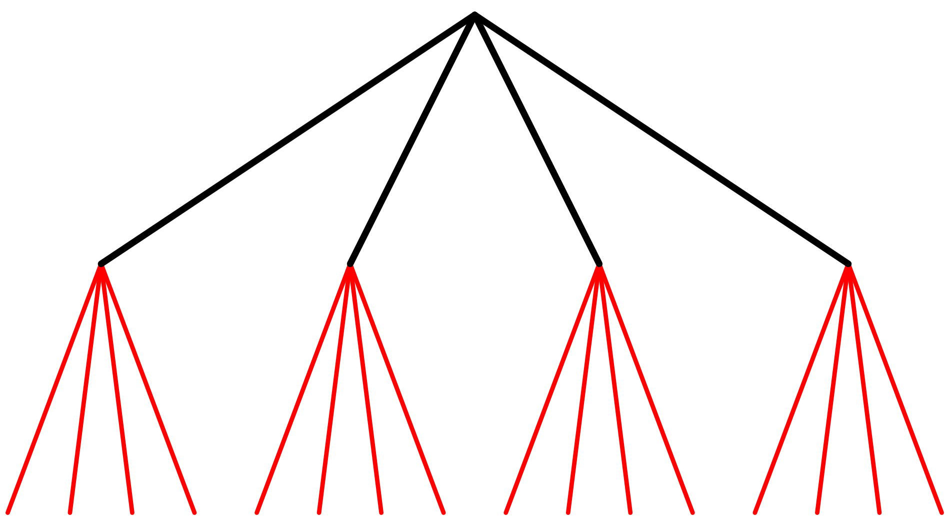
- Complexity: $O(N^2\,(\log N)^2)$ for N^2 pixels.



Results

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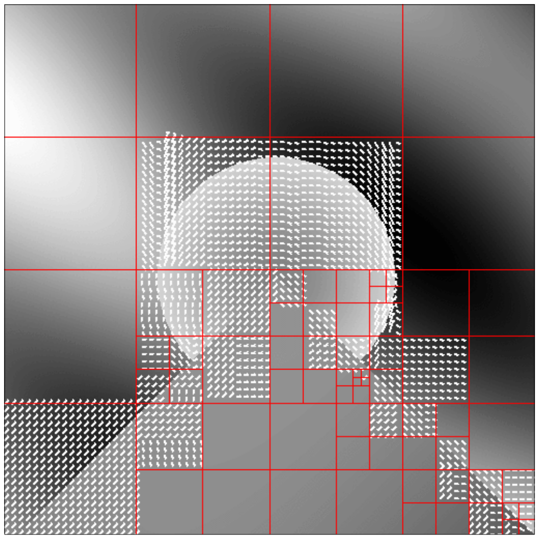
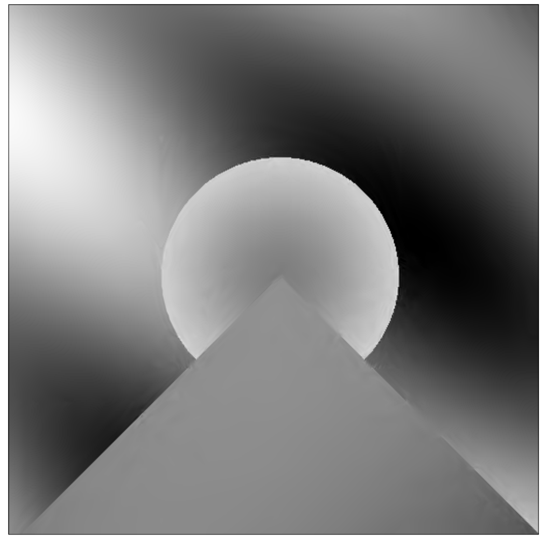
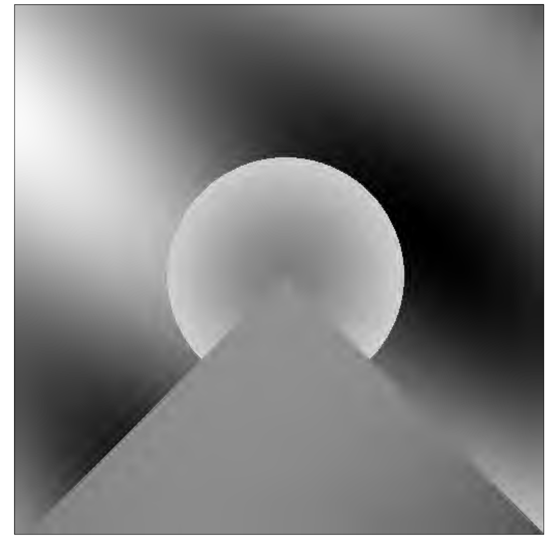
M=2650



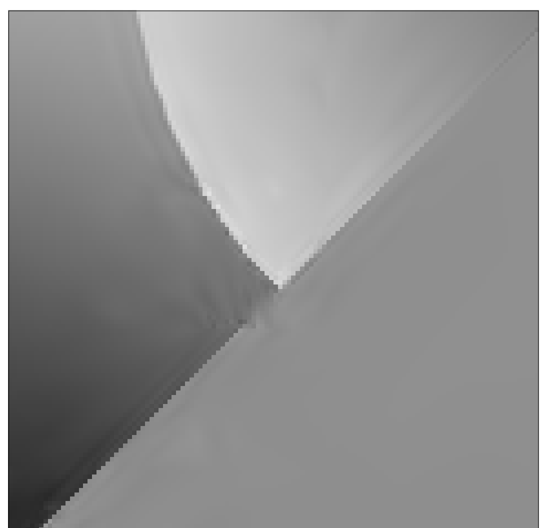
PSNR $= 45{,}97\,$dB



PSNR $= 40{,}17\,$dB



Bandelets



Wavelets

Theorem

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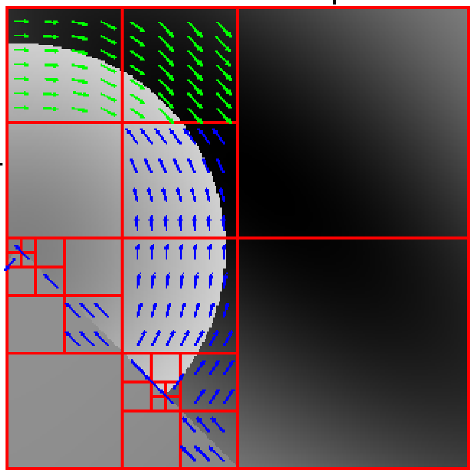
$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



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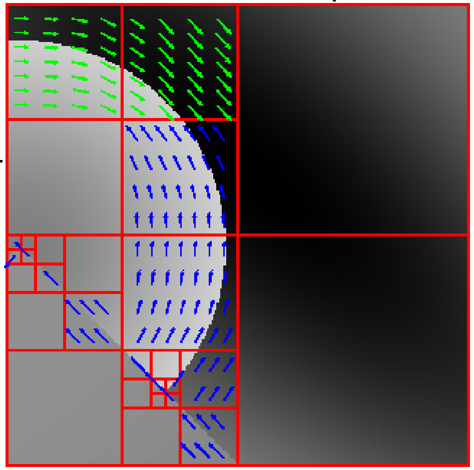

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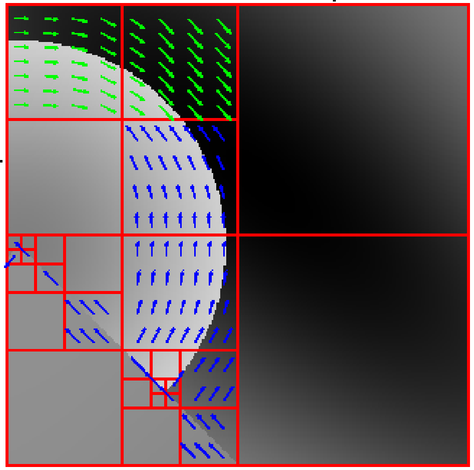

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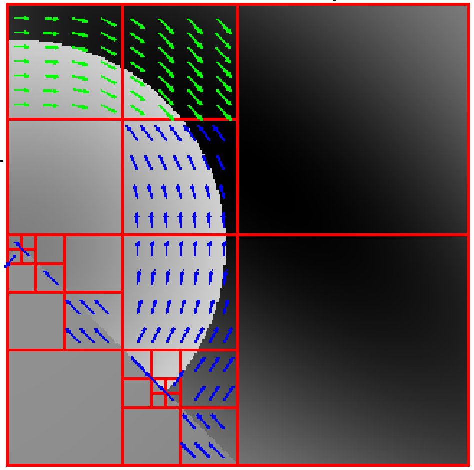

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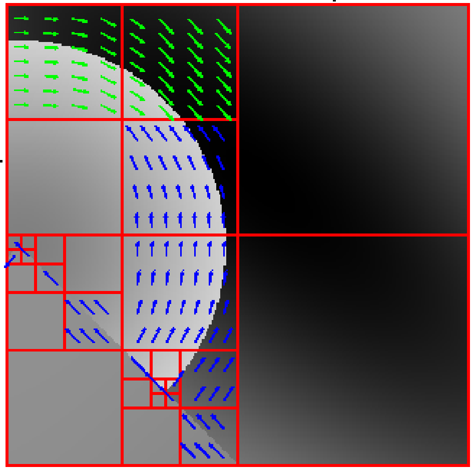

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$$\|f - f_M\|^2 \leqslant C\, (\log M)^3\, M^{-2} \quad .$$

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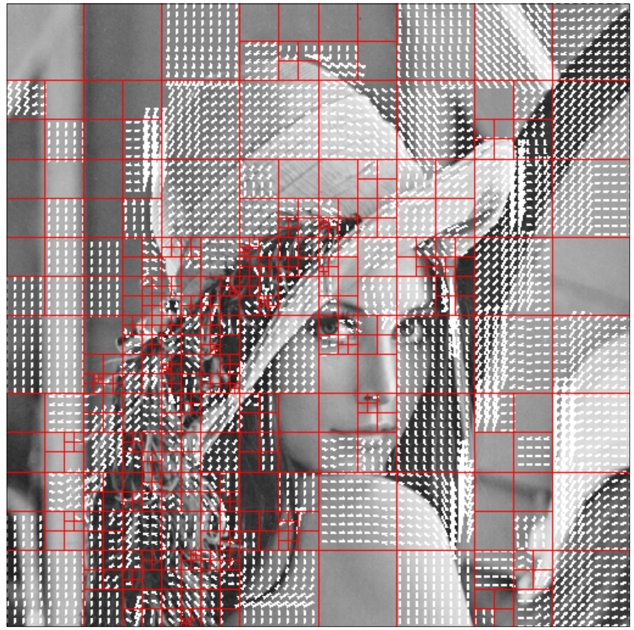
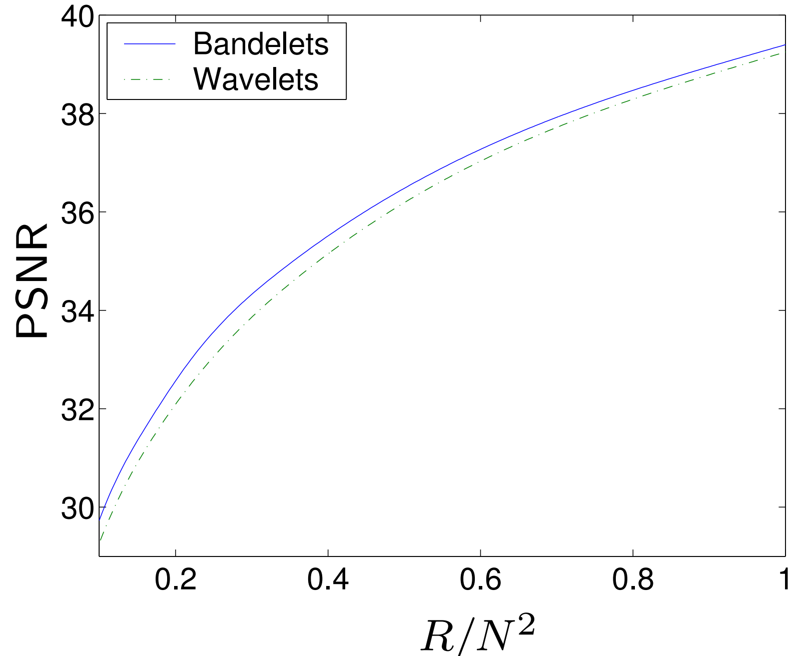
- $O(N^2 (\log_2 N)^2)$ operations.

Original



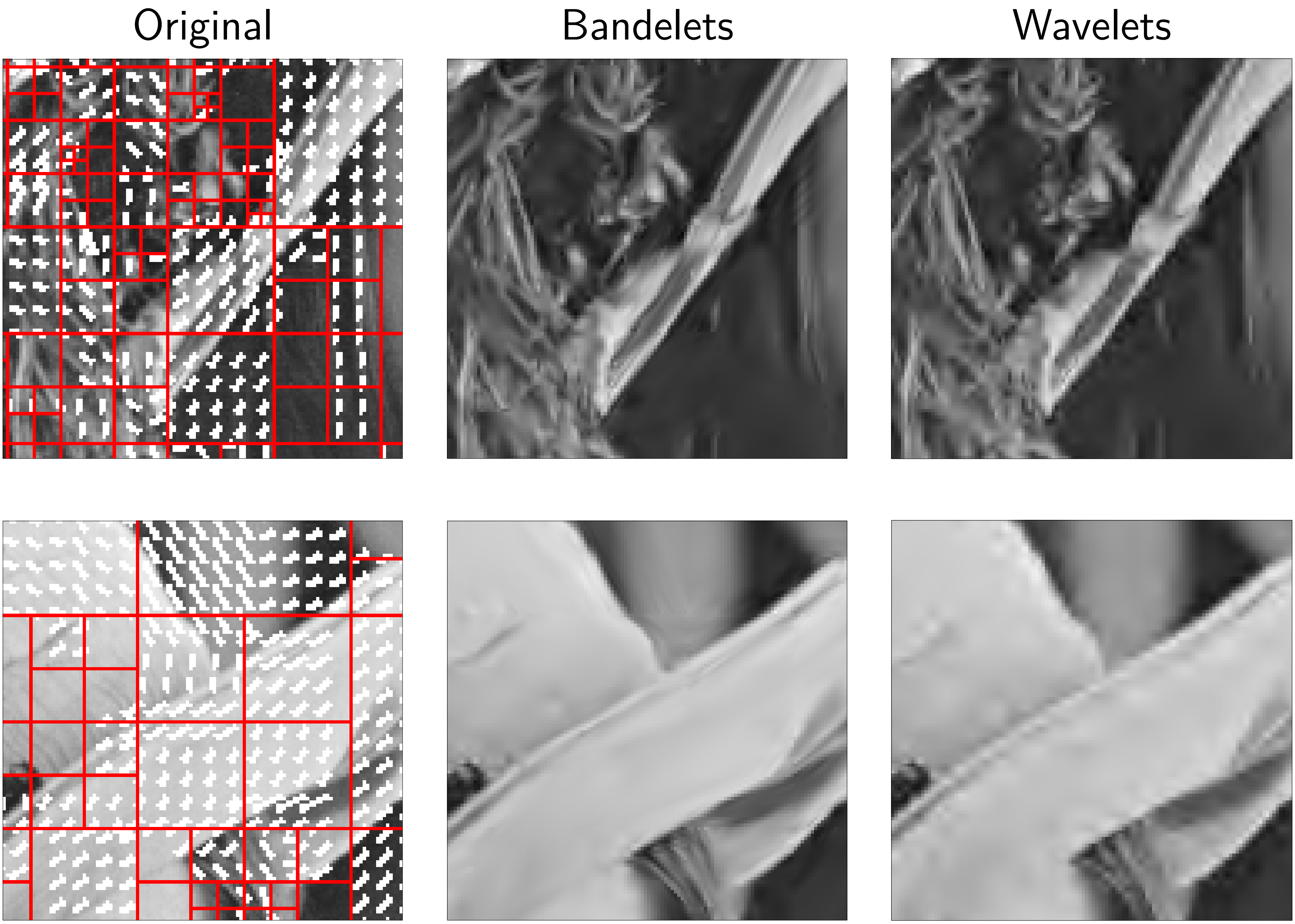
Distortion-Rate



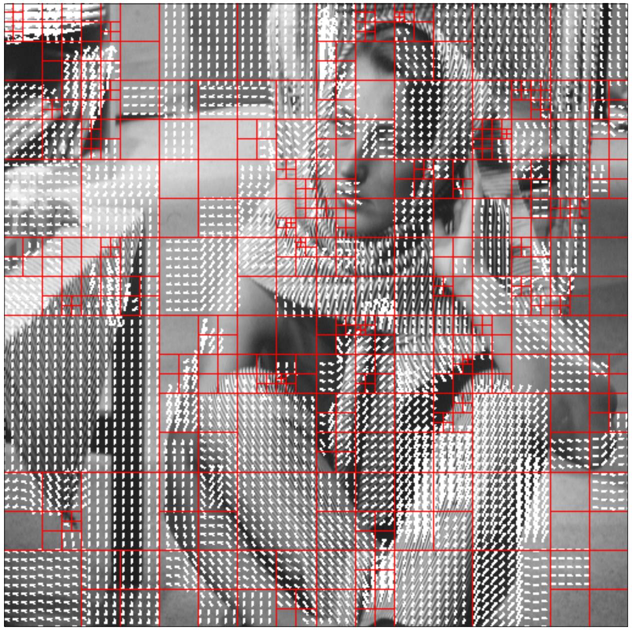
$R/N^2 = 0.22$ bpp

Bandelets (33.05 db)

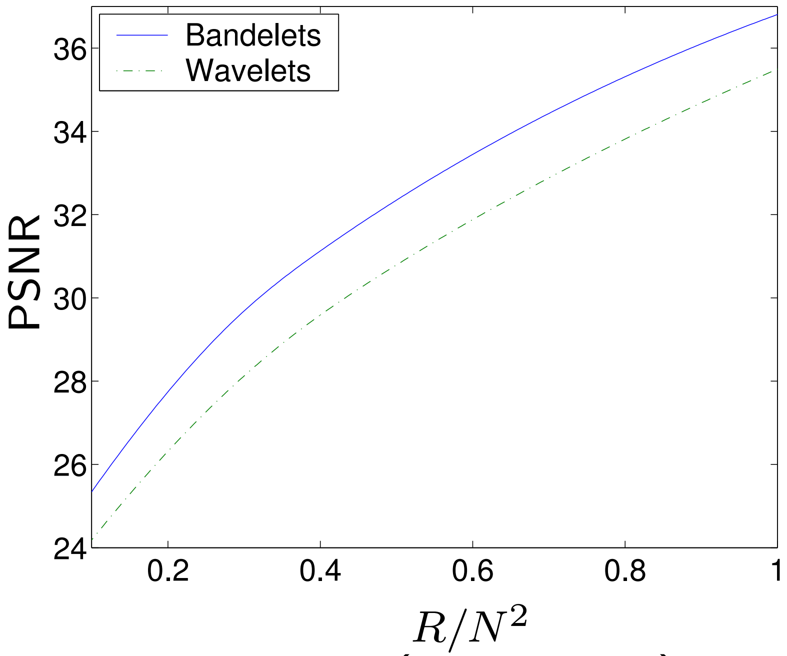
Wavelets (32.54 db)


Original
Bandelets
Wavelets

Original



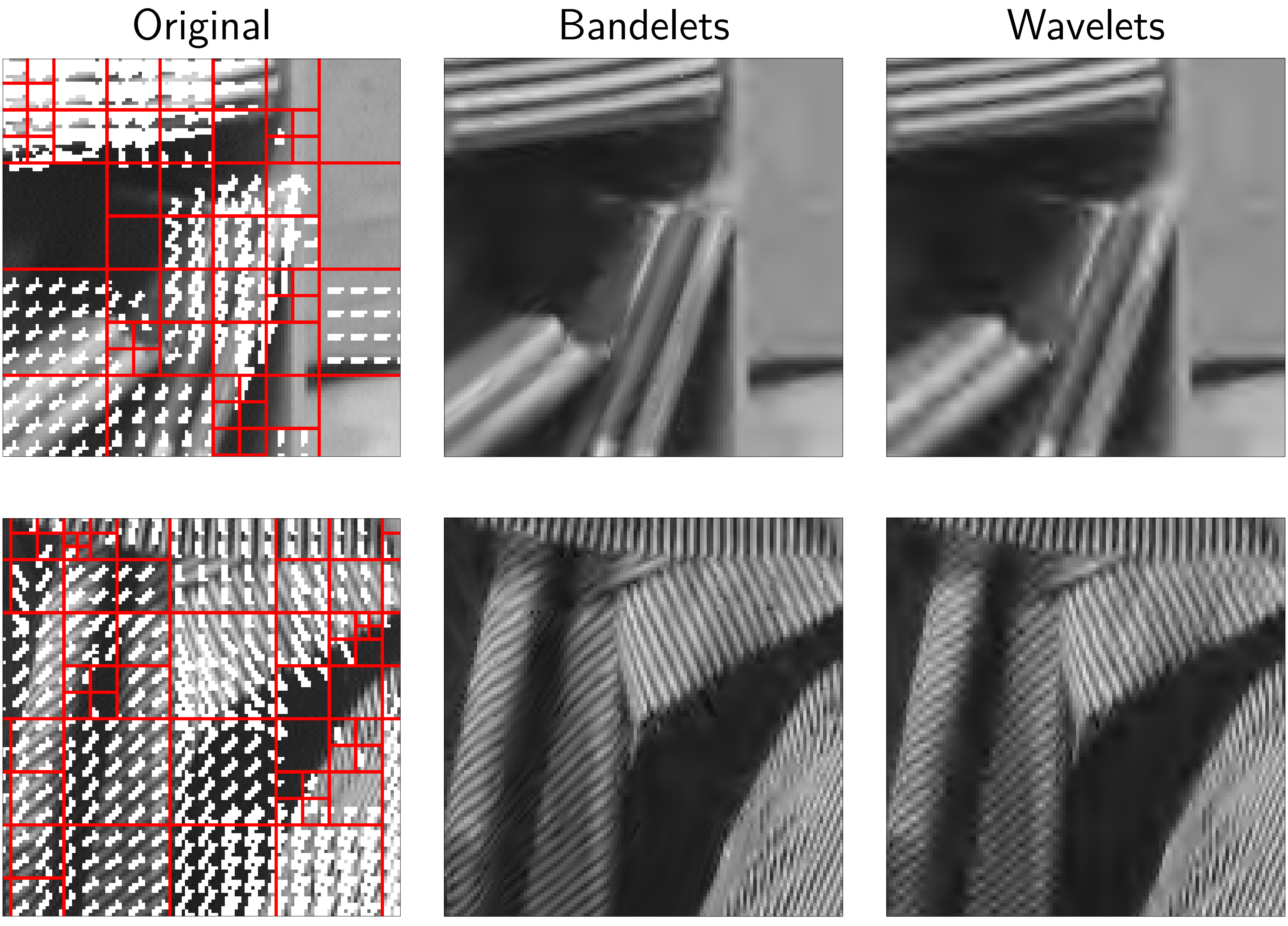
Distortion-Rate



$R/N^2 = 0.40$ bpp

Bandelets (31.22 db)

Wavelets (29.68 db)



Original
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Denoising

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$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

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- Design of a penalized estimator.

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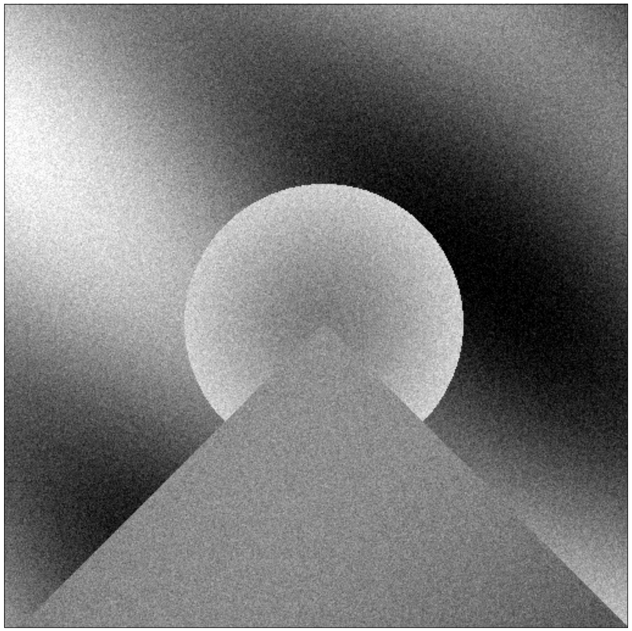
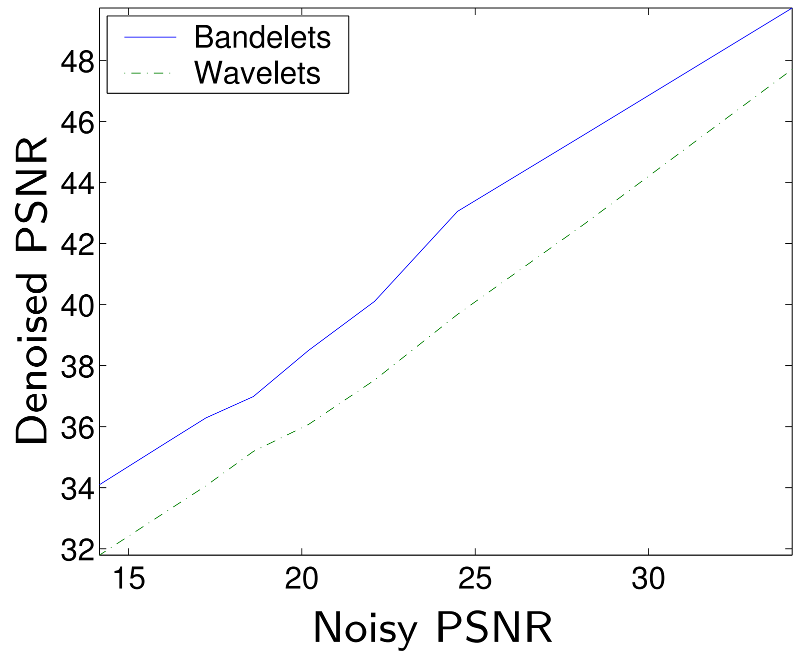
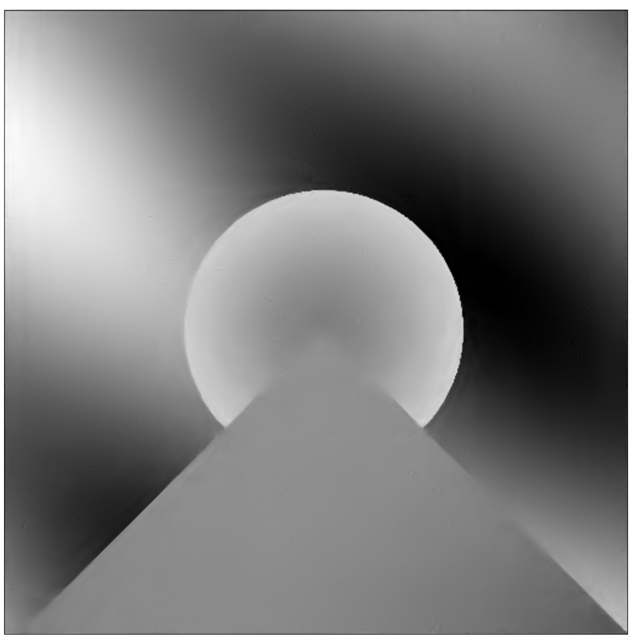
- Allows to reuse the compression algorithm almost directly.
- No theoretical results but a practical algorithm with a flow estimation.

Noisy $(20.19\,\mathrm{dB})$

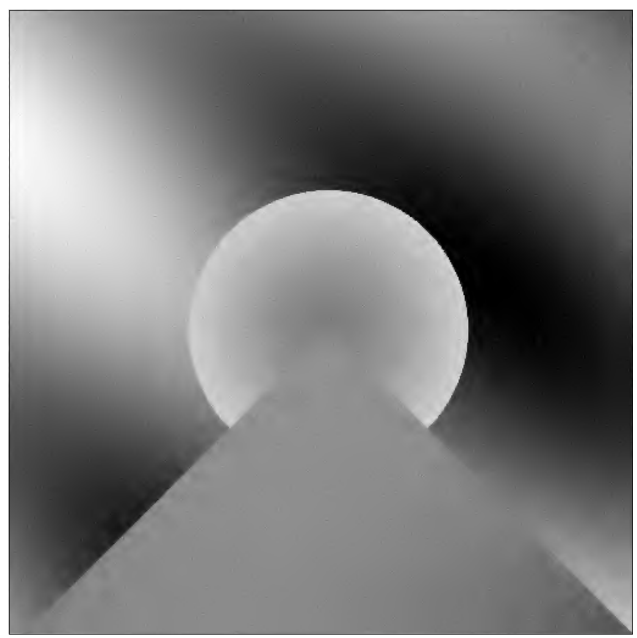




Bandelets $(30.29\,\mathrm{dB})$



Wavelets $(28.21\,\mathrm{dB})$



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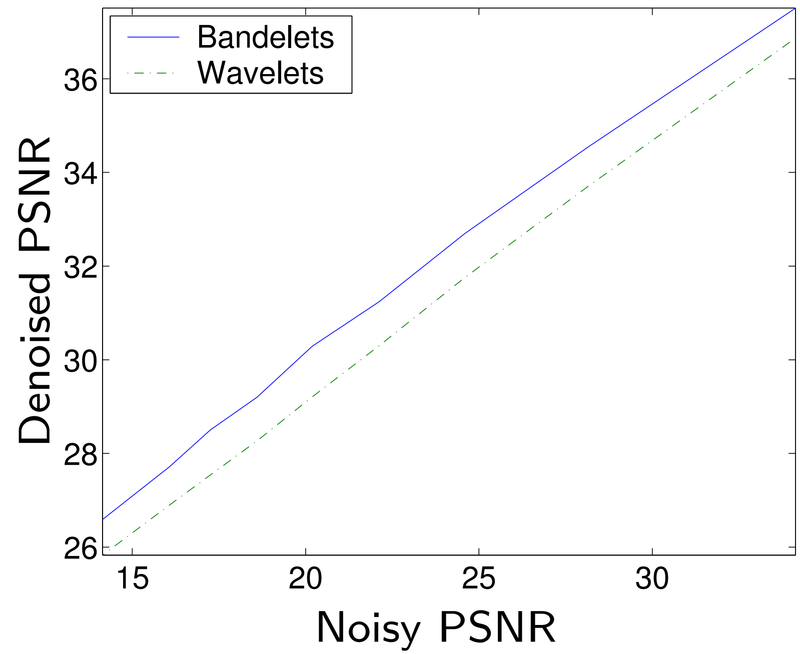

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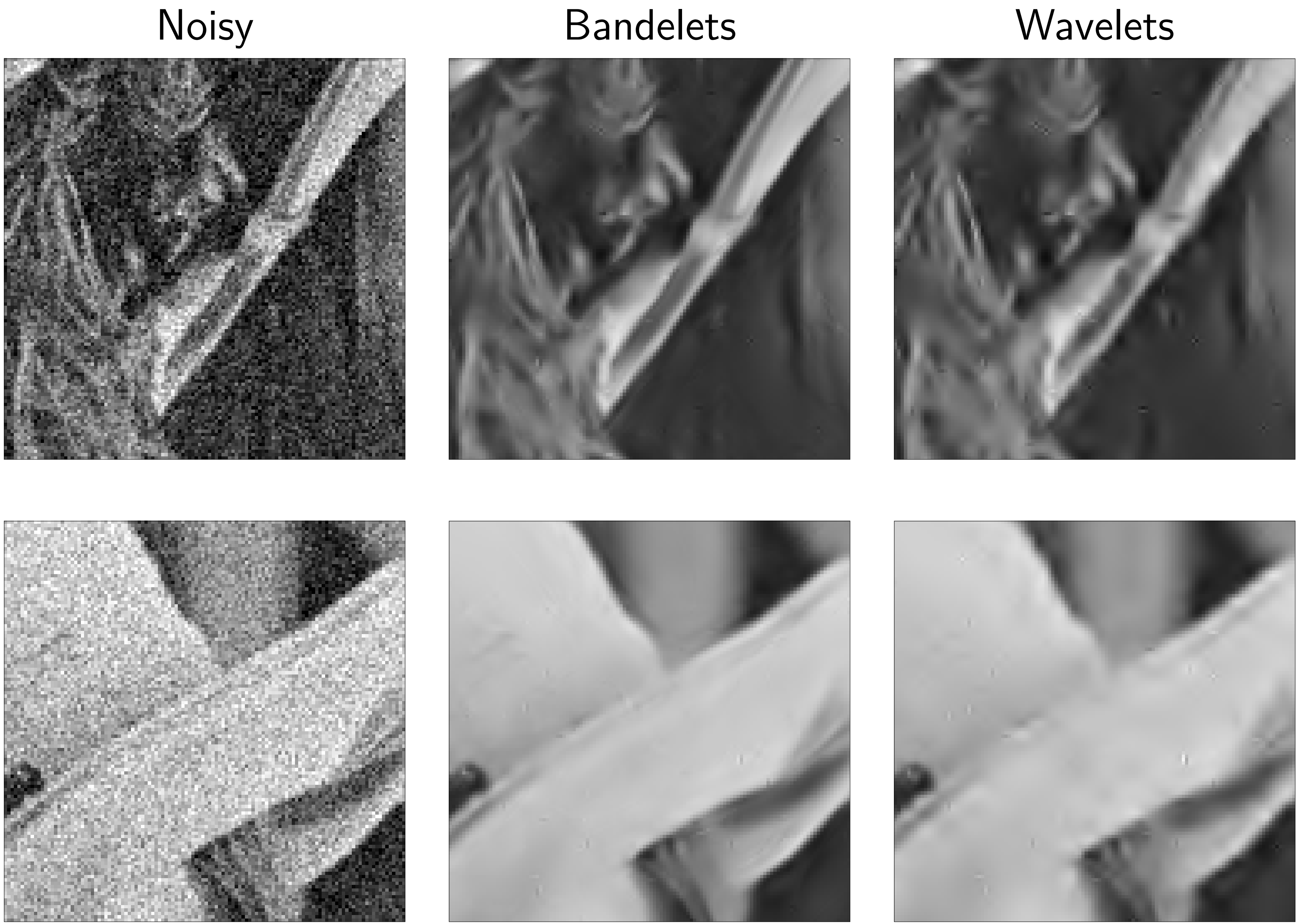
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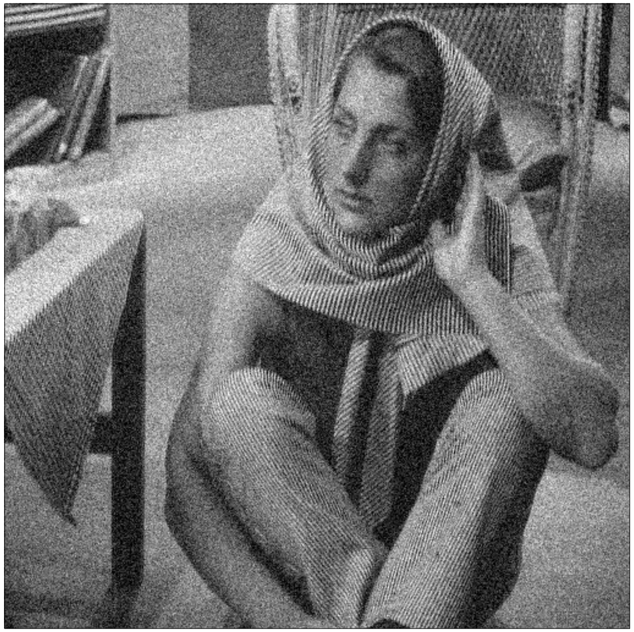
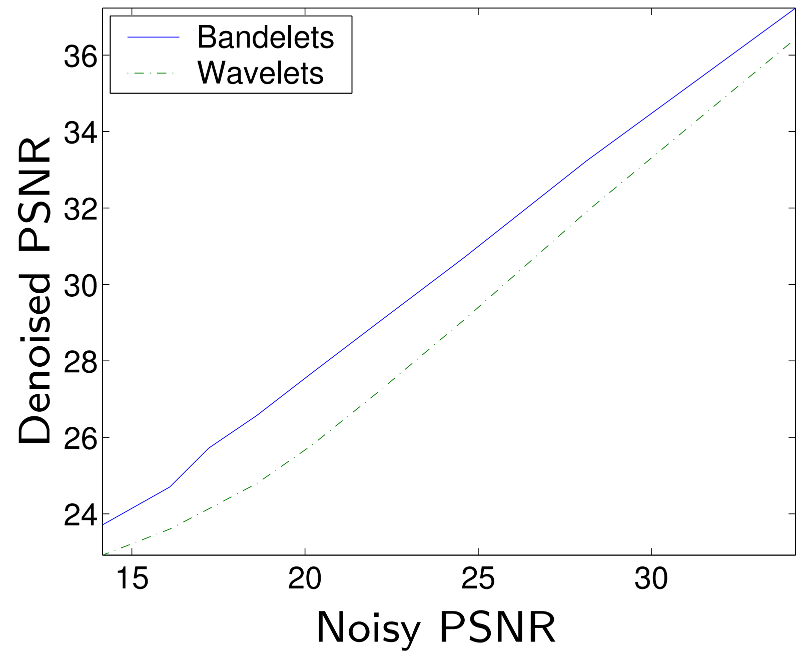
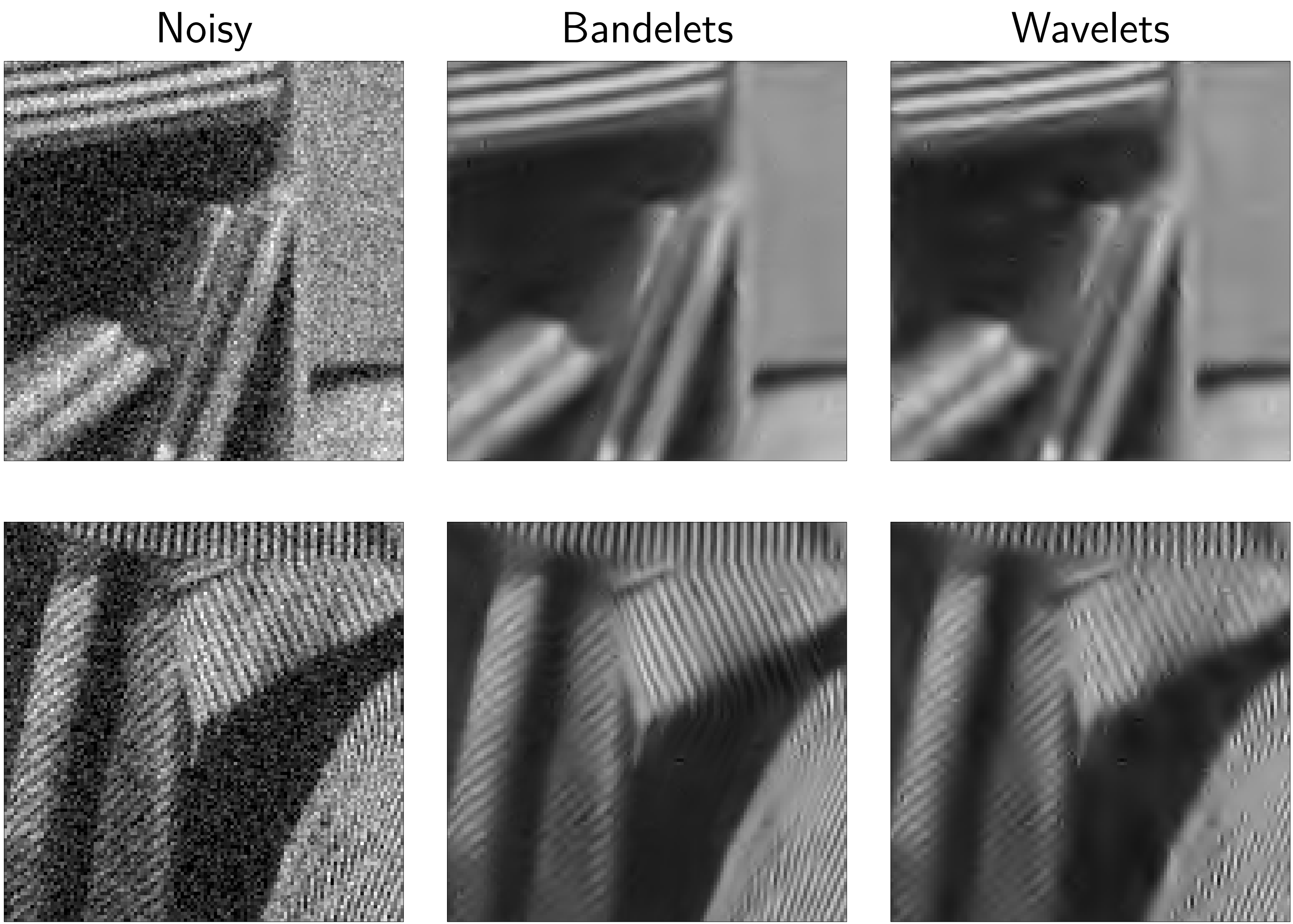




Bandelets $(27.68\,\text{dB})$

Wavelets $(25.79\,\text{dB})$


Noisy
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Penalized Model Selection

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gives an almost optimal result on the estimator risk.

Theoretical Result

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- Sampling case: $Y(x_i) = f(x_i) + \epsilon(x_i)$ with ϵ a white noise of variance σ^2:

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- **More details on the true bandelets construction next week**.