Geometrical Image Compression with Bandelets

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- Need to take advantage of geometrical image regularity to improve representations.
- Second generation image coding dream: a bridge between *Image Processing* and *Computer Vision*.
- Building harmonic analysis representations adapted to complex *geometry*.

Edge Detection

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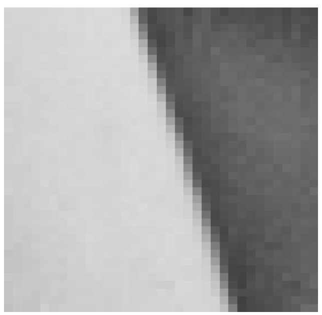
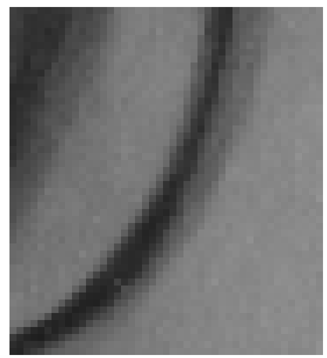
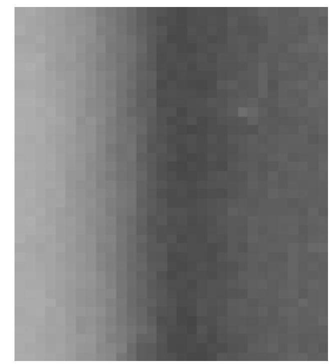
- Edges are blurred singularities.

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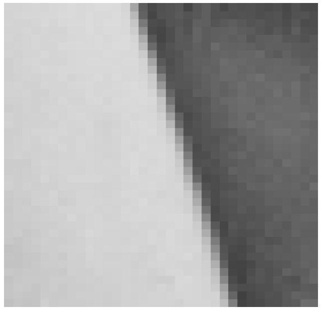

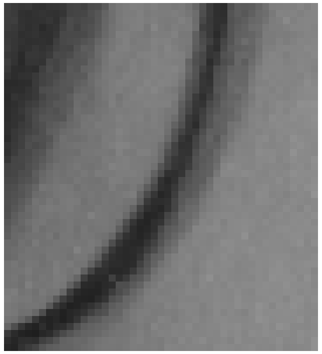

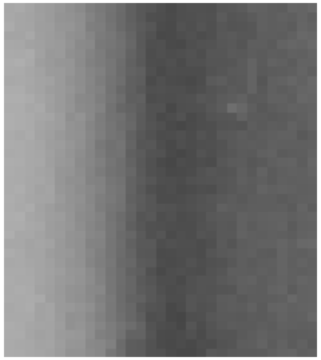

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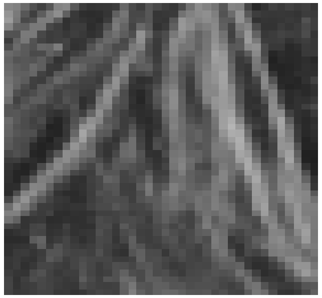

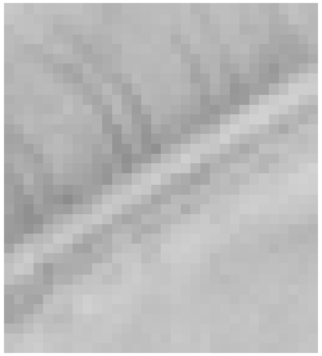

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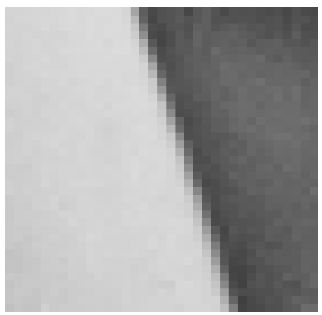
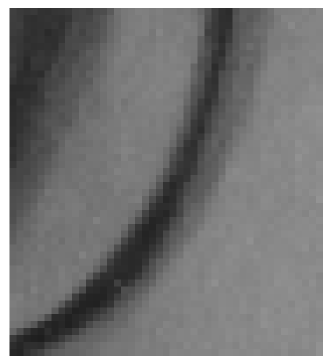
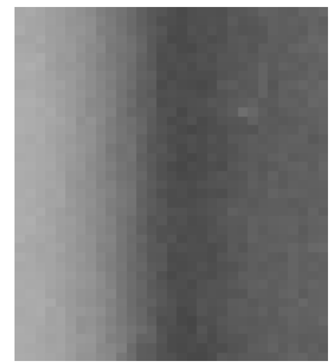

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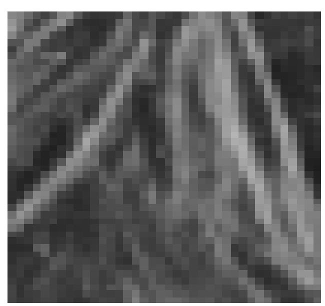
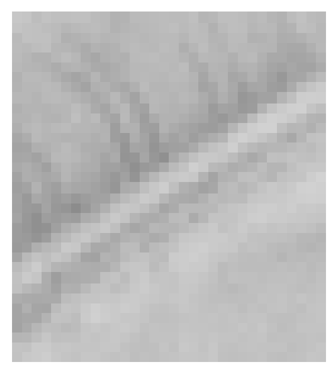

- How can the estimation of geometry become well-posed ?

Overview

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- Geometric flow and bandelet bases.
- Approximations in bandelet bases.
- Image compression.

Sparse Representation in a Basis

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- Decomposition in an orthonormal basis $\mathcal{B} = \{g_m\}_{m \in \mathbb{N}}$

$$f = \sum_{m \in \mathbb{N}} \langle f, g_m \rangle \, g_m \ .$$

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  select the M largest inner products:

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- **Problem** : How to choose the basis $\mathcal{B}$ so that

$$\|f - f_M\|^2 \leqslant C M^{-\alpha} \text{ with } \alpha \text{ large ?}$$

Separable 2D Wavelet Basis

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- The family

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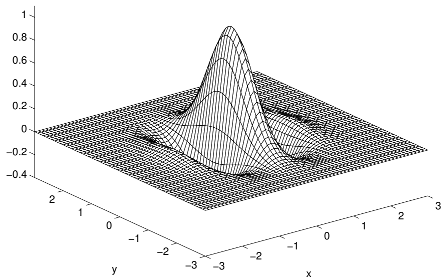

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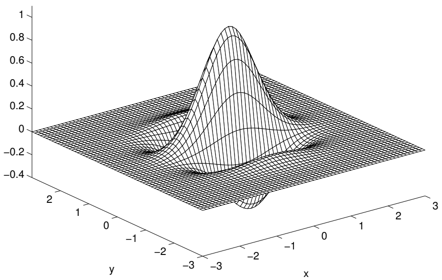

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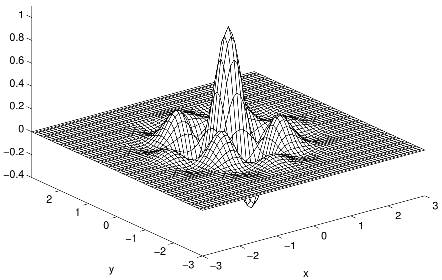

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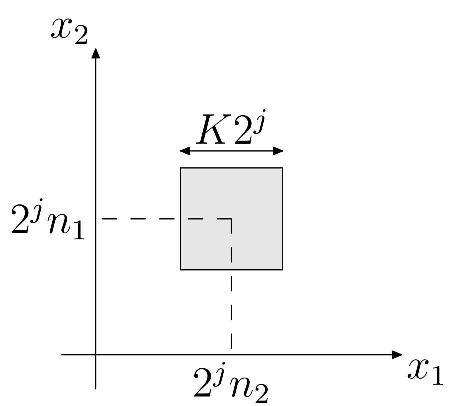

Isotropic Wavelets Support

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- Images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

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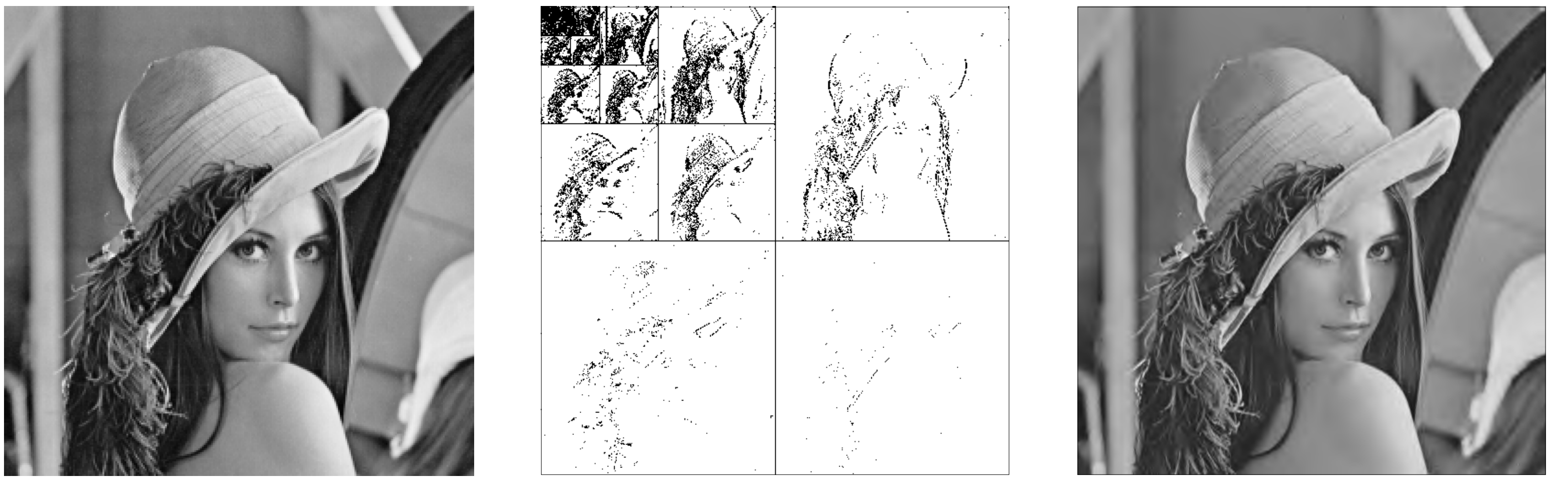

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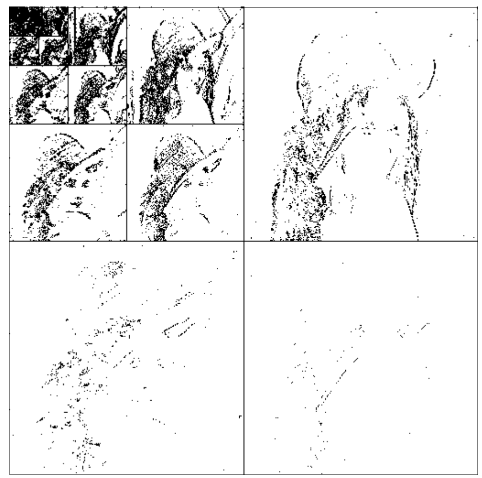

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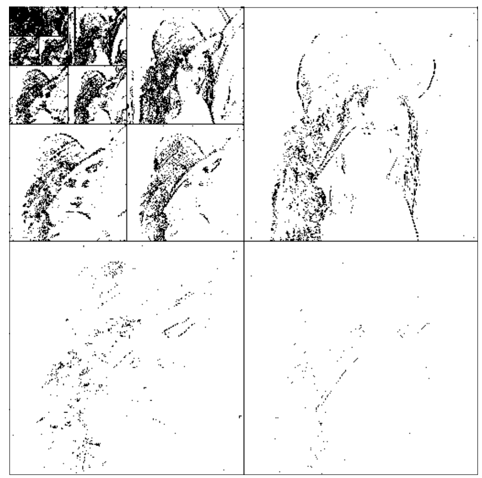

f_M

- *(Cohen, DeVore, Petrushev, Xue)*: Optimal for bounded variation functions: $\|f - f_M\|^2 \leqslant C\,\|f\|_{TV}\,M^{-1}$.
- But: does not take advantage of any geometric regularity.

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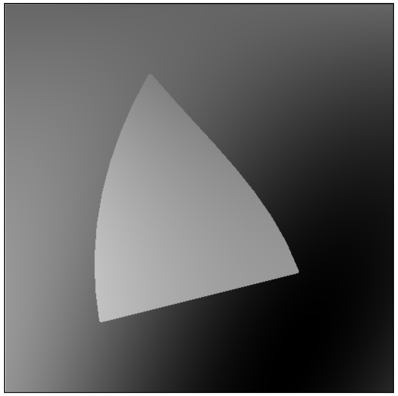
- Approximations of f which is $\mathbf{C}^\alpha$ away from "edges" which are piecewise $\mathbf{C}^\alpha$ curves $(\alpha \geqslant 2)$:



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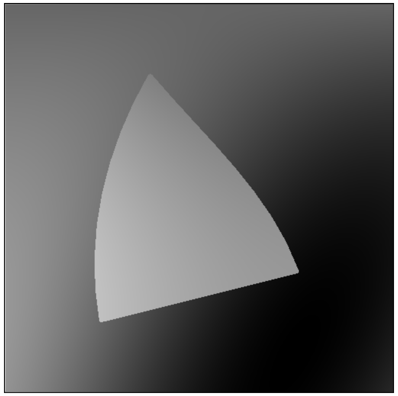
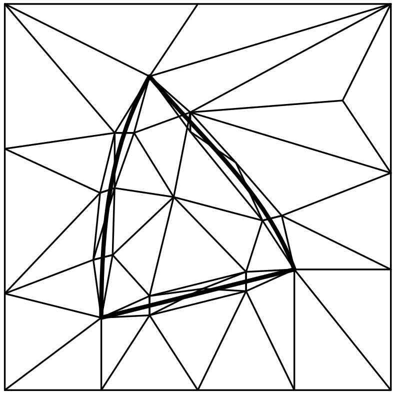
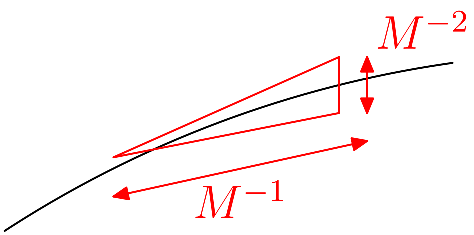
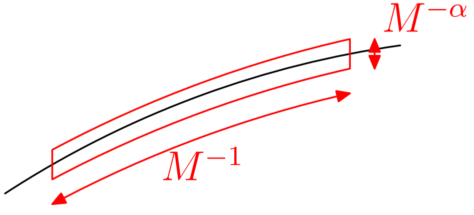




  - with M wavelets : $\|f - f_M\|^2 \leqslant C\, M^{-1}$,
  - with M triangles : $\|f - f_M\|^2 \leqslant C\, M^{-2}$,
  - with M curvelets (*Candes, Donoho*) : $\|f - f_M\|^2 \leqslant C\,(\log M)^3\, M^{-2}$,
  - other approaches: (*Cohen, Matei*), (*Kingsbury*), (*Baraniuk*), (*Dragotti, Vetterli, Do*)...
  - with M higher order geometric elements : $\|f - f_M\|^2 \leqslant C\, M^{-\alpha}$





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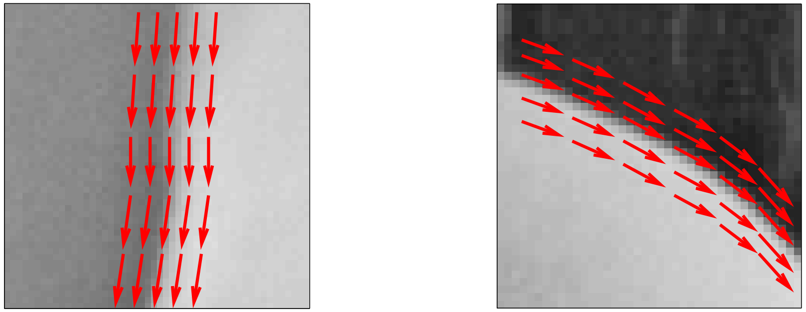

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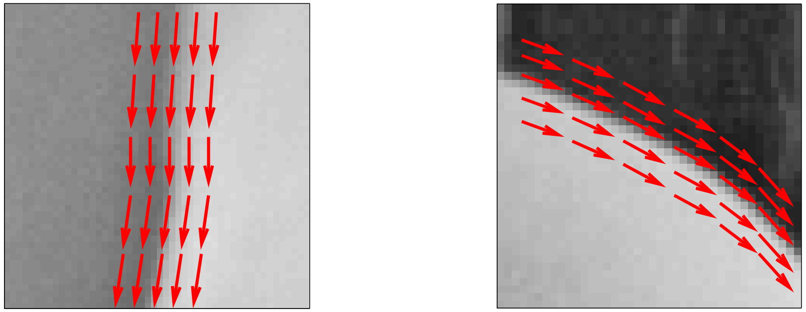
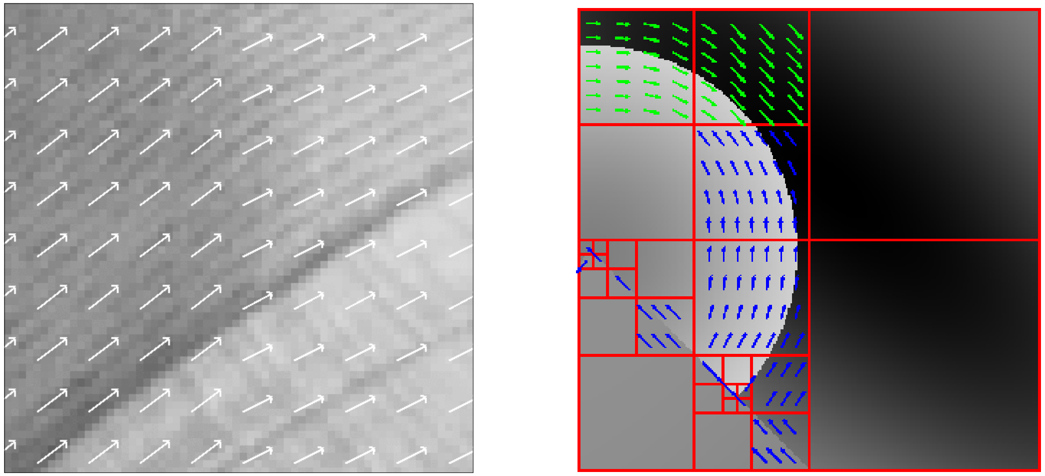


- The image is divided in multiple regions where the flow is parallel:



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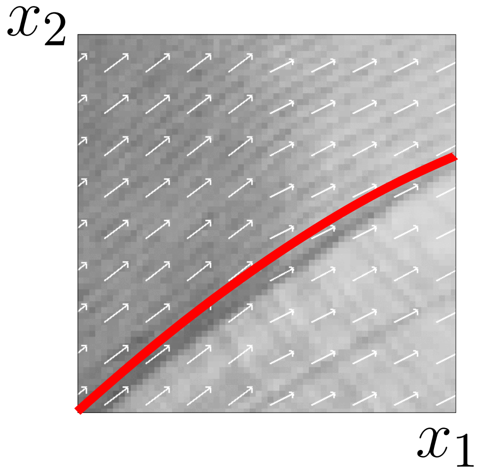
$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1))\ .$$



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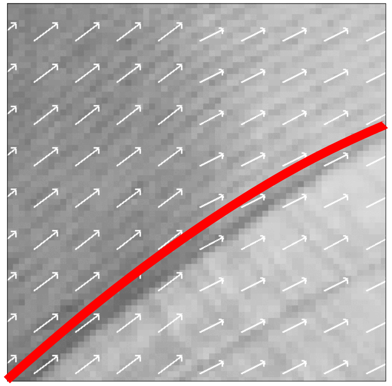

$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\, \mathrm{d}u$$

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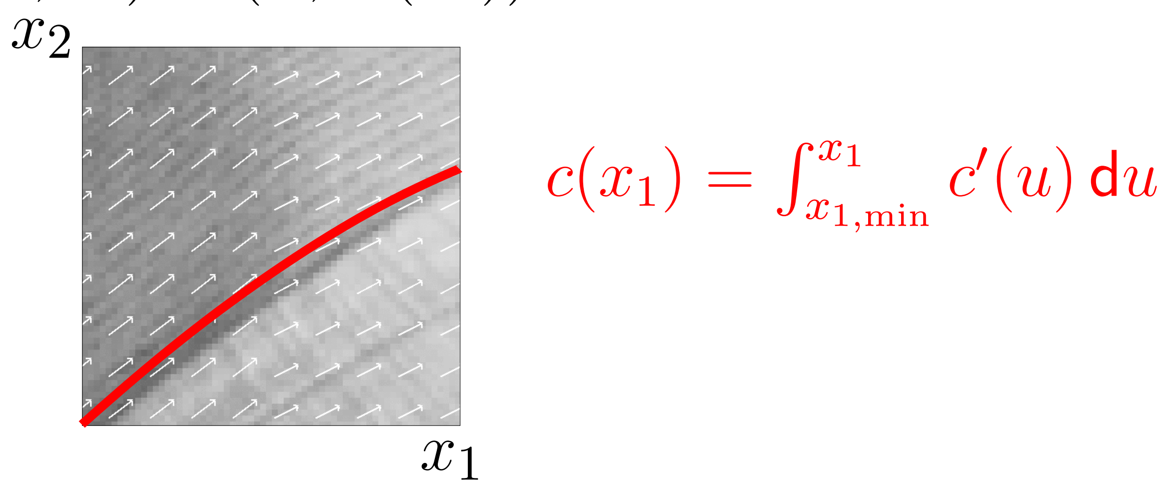

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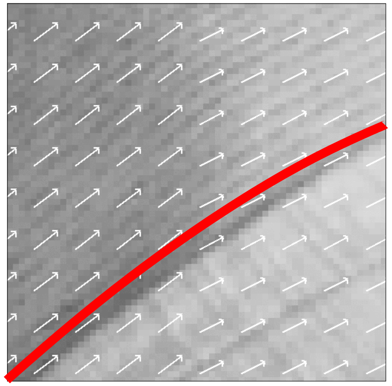

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- We thus decompose f in a *warped wavelet basis* of $\mathbf{L^2}(\Omega)$:

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- Bandeletization: replaces $\{\phi_{j,m_1}(x_1)\}_{m_1}$ by a wavelet family $\{\psi_{l,m_1}(x_1)\}_{l>j\,,m_1}$ that generates the same space.
- Warped wavelet basis of $\mathbf{L^2}(\Omega)$:

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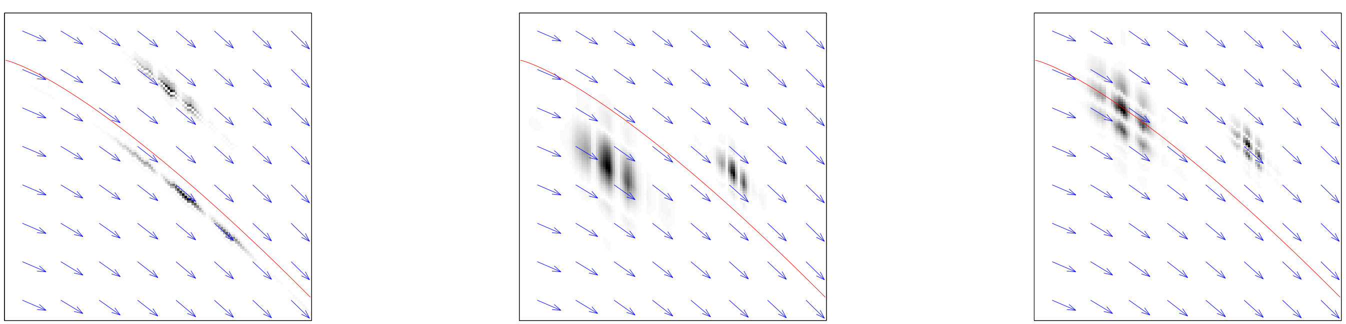
Anisotropic Isotropic



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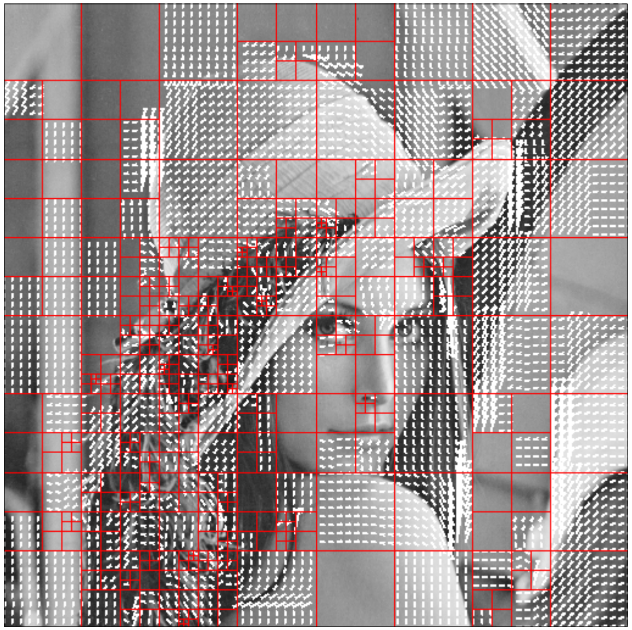

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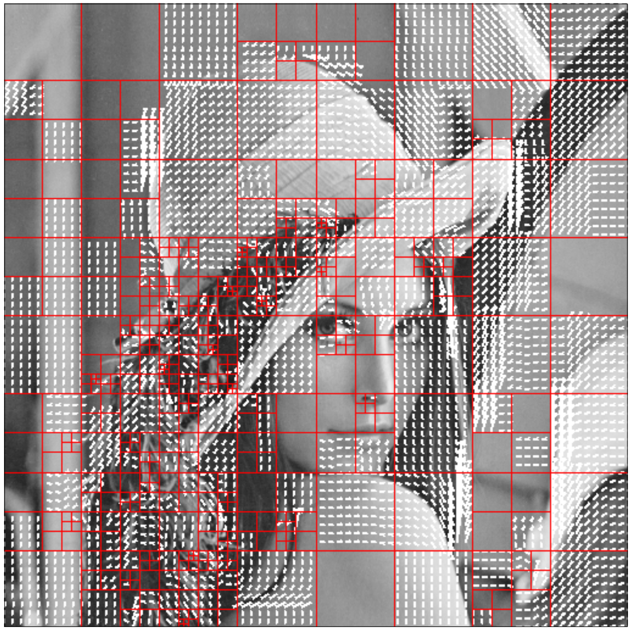

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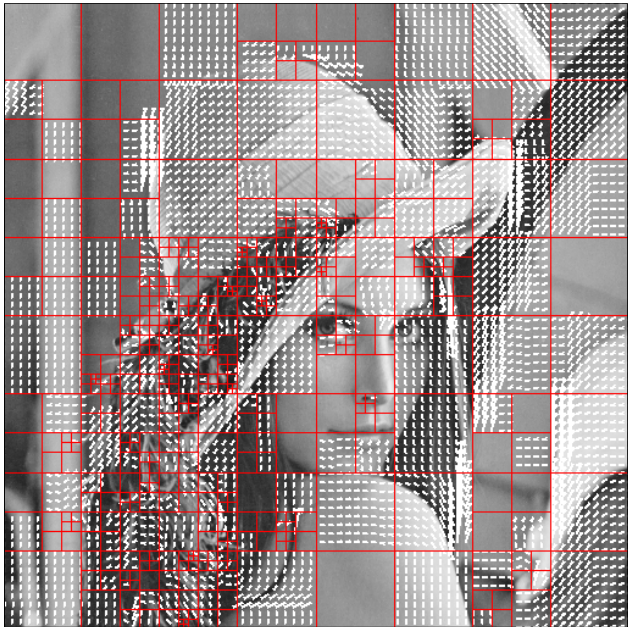

- Fast bandelet transform $(O(N^2))$:
  - resampling, warped wavelet transform, bandeletization.
- No discontinuities at boundaries with an adapted lifting scheme.

Computation of a Parameterized Flow

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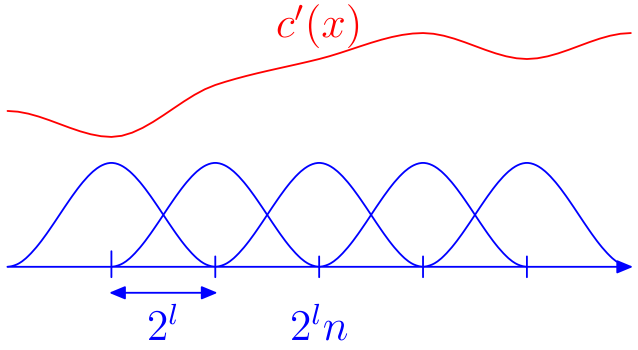
- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by:

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$



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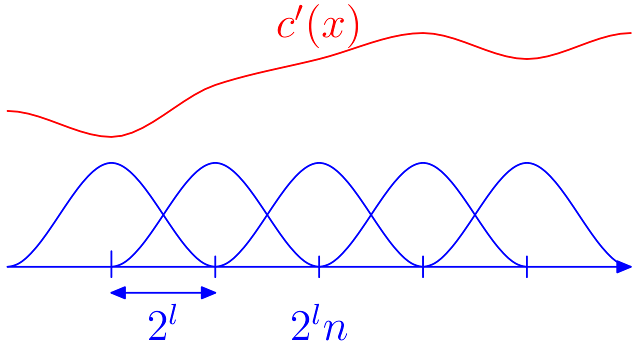
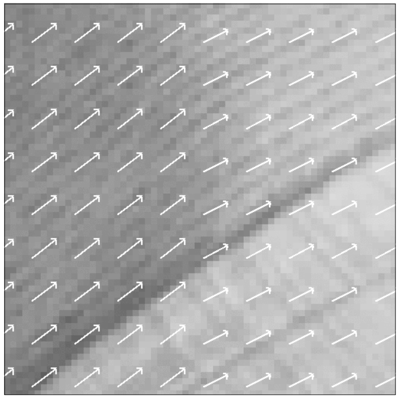


by finding the $L\,2^{-l}$ parameters α_n that minimize

$$\int_\Omega \left| \vec{\nabla} f(x_1, x_2) \,.\, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \,.$$



Scale Adaptation of the Parameterized Flow

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- Adapting the scale 2^l to the signal regularity along the flow:

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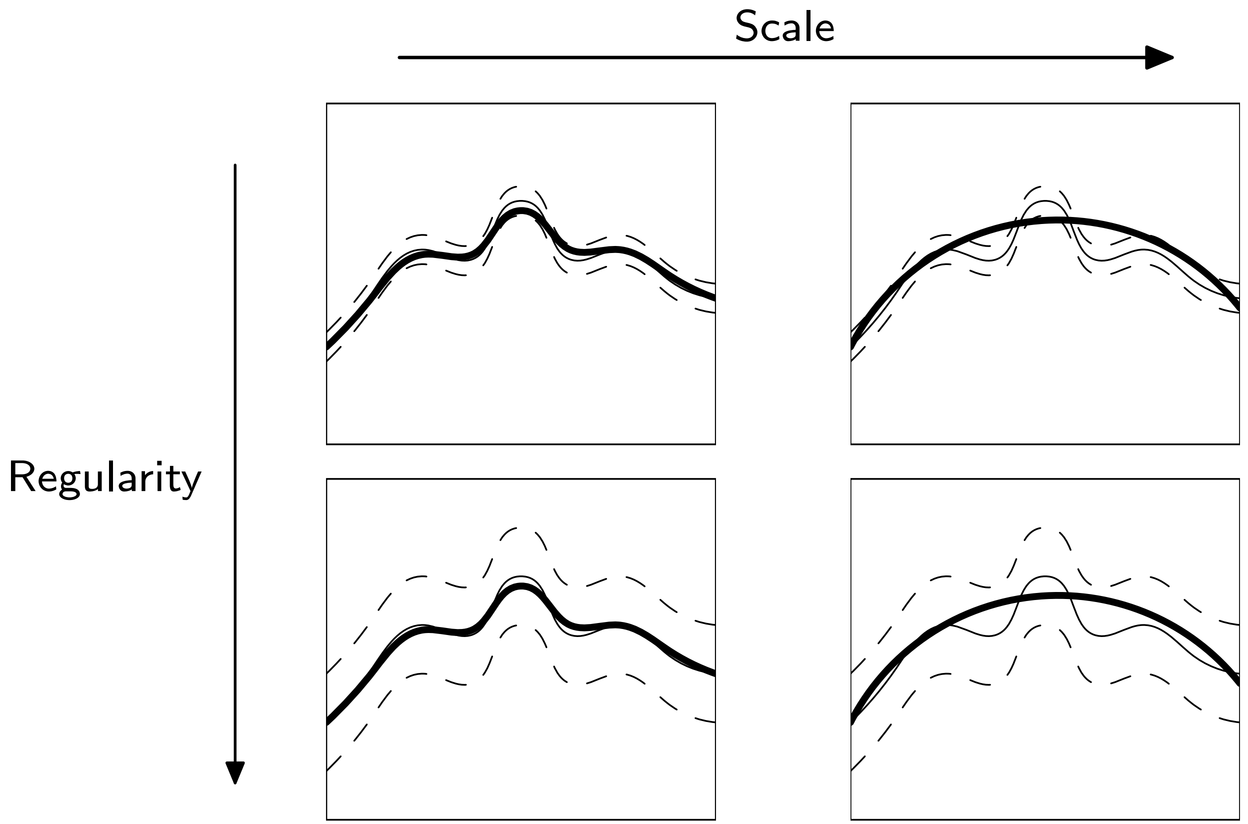

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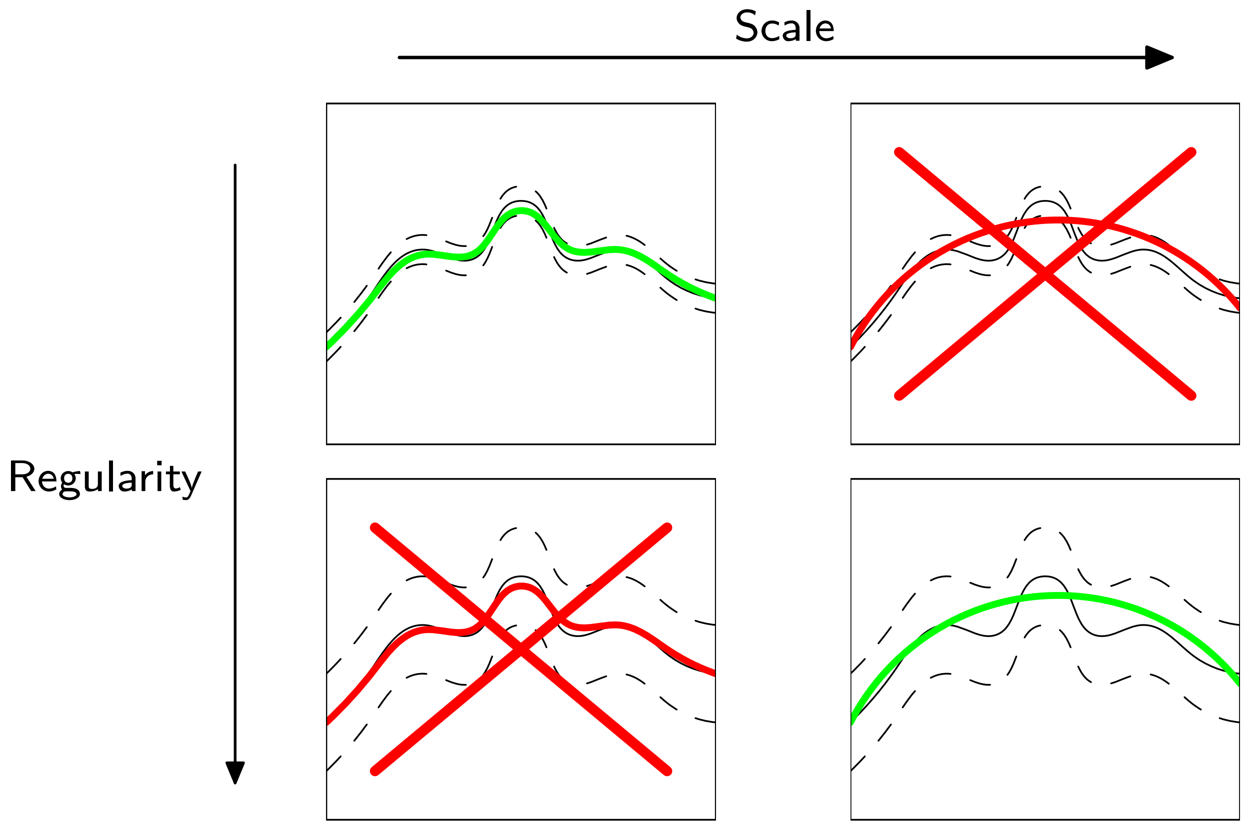

M-Term Approximation

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- A bandelet image approximation f_M is specified by:
  - a dyadic square image segmentation, represented by the M_s inside nodes of the segmentation quad-tree,
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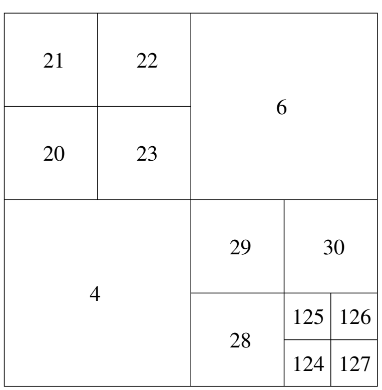
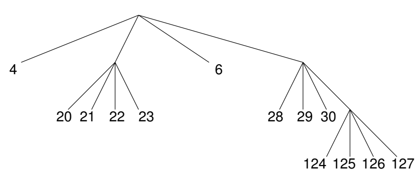
- Total number of parameters:

$$M = M_s + \sum_i \Big(M_{g,i} + M_{b,i} \Big) \ .$$





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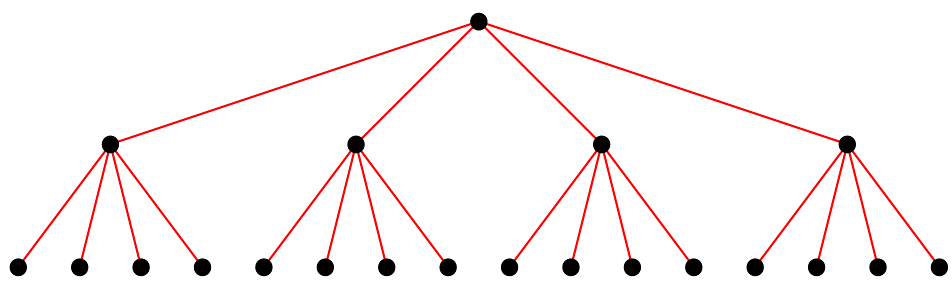
- Computational complexity: $O(N^2\,(\log N)^2)$ for N^2 pixels.



Approximation Theorem

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- **Theorem:** Suppose that $\tilde{f}$ is $\mathbf{C}^\alpha$ away from "edges" that are piecewise $\mathbf{C}^\alpha$ non tangent curves.
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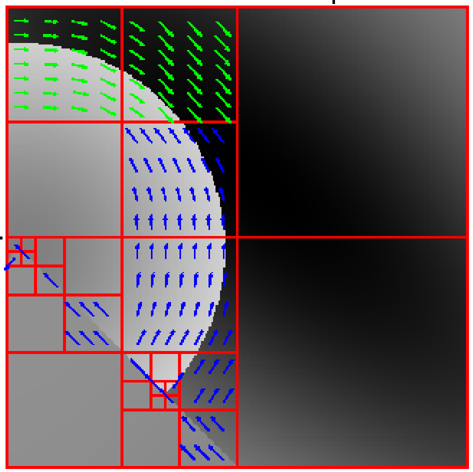

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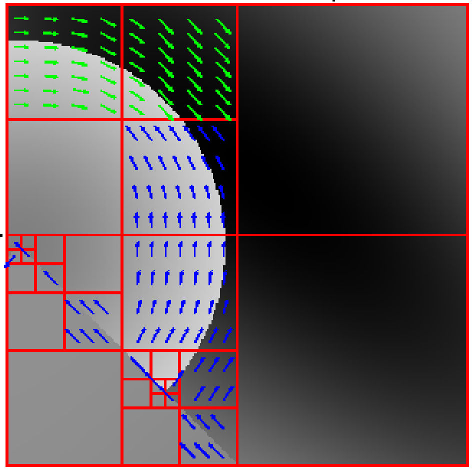

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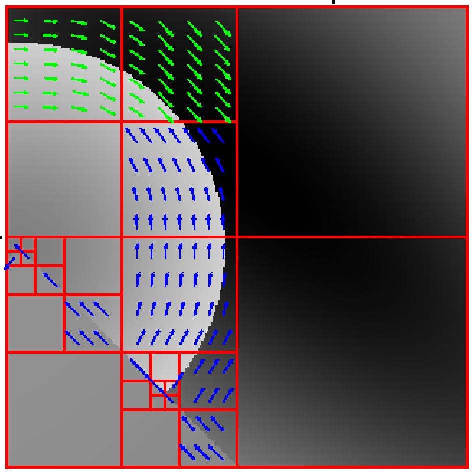

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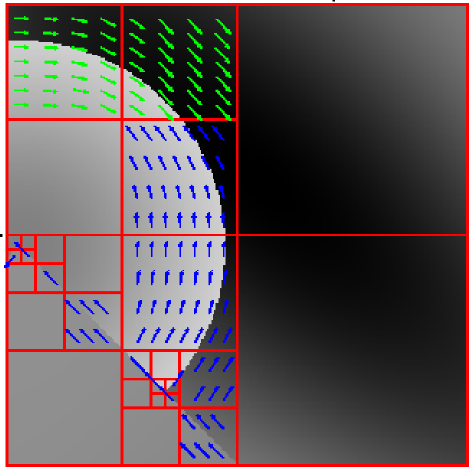

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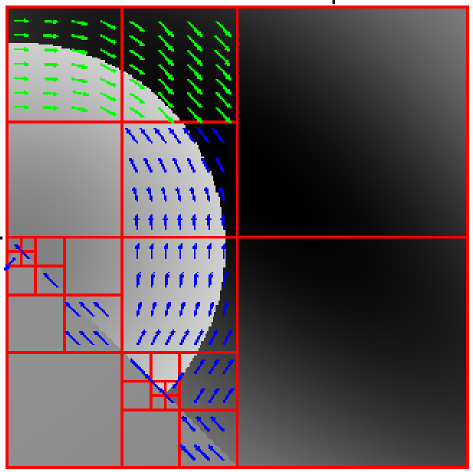

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- Improvement over curvelets for which $\|f - f_M\|^2 \leqslant C\, (\log_2 M)^3\, M^{-2}$.

Piece-wise Regular Approximation

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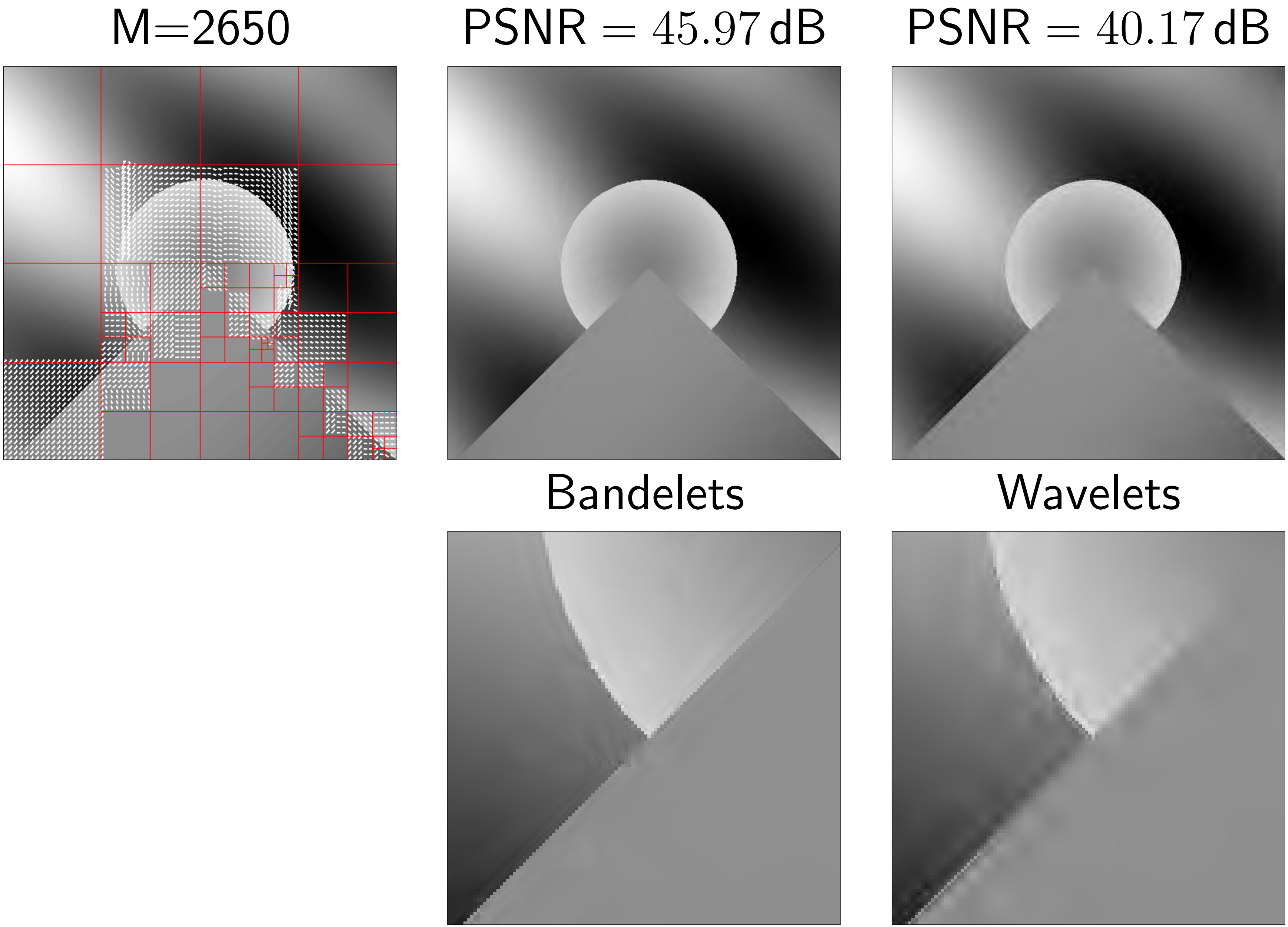

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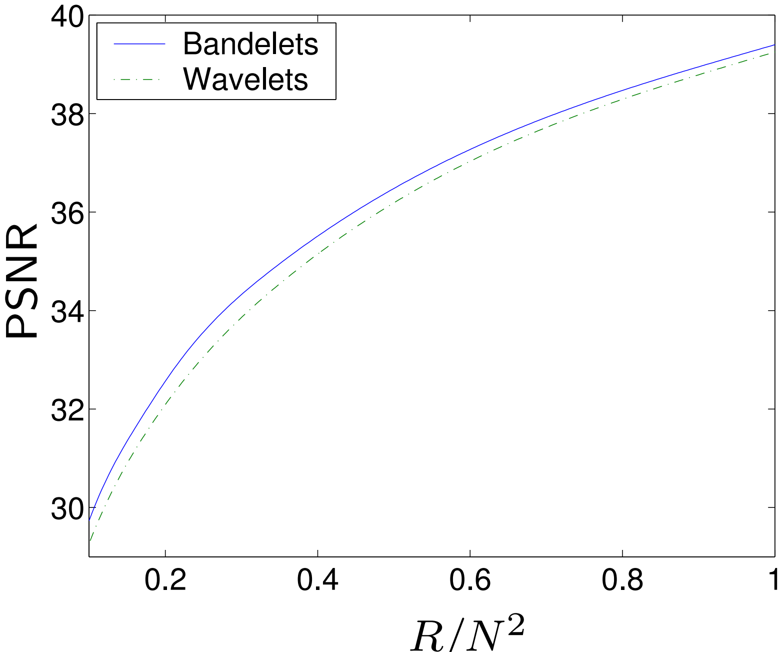
Original

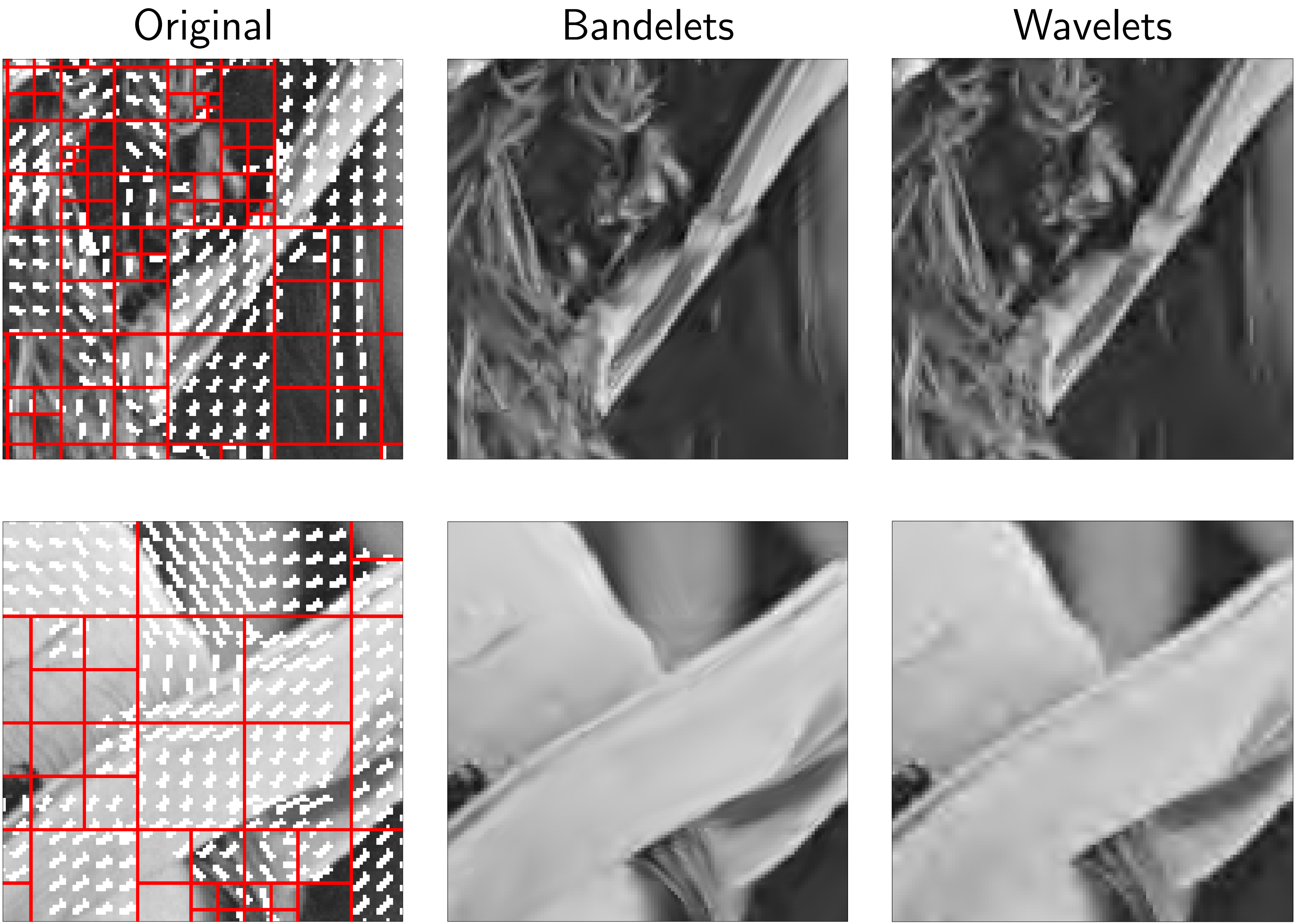
Distortion-Rate



$R/N^2 = 0.22$ bpp

Bandelets $(33.05$ db$)$

Wavelets $(32.54$ db$)$


Original
Bandelets
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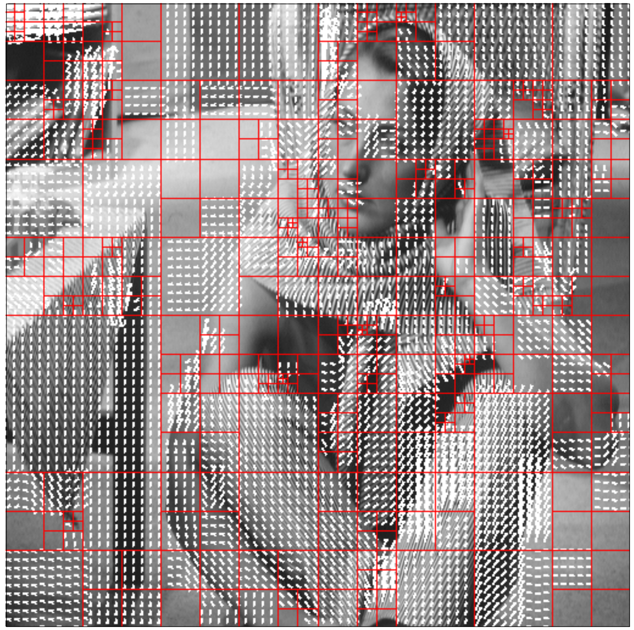

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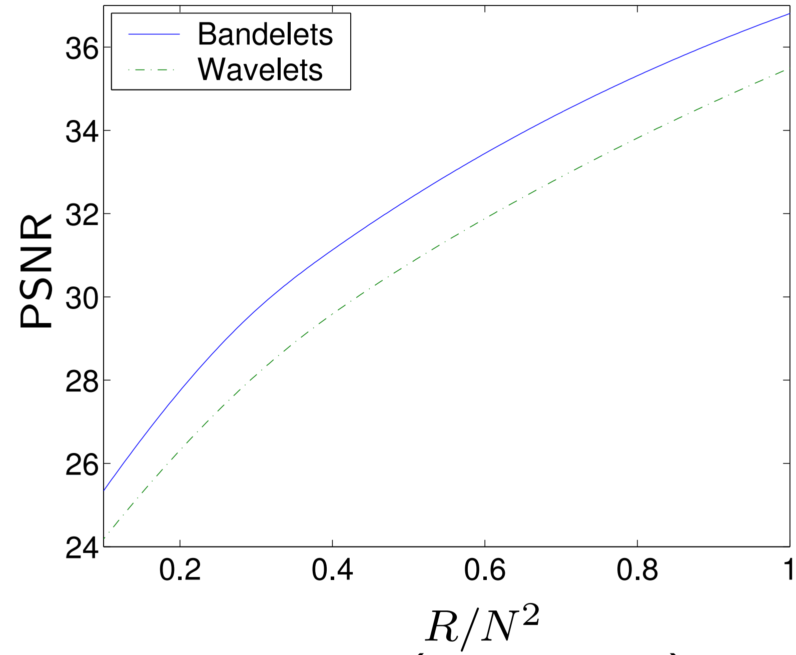
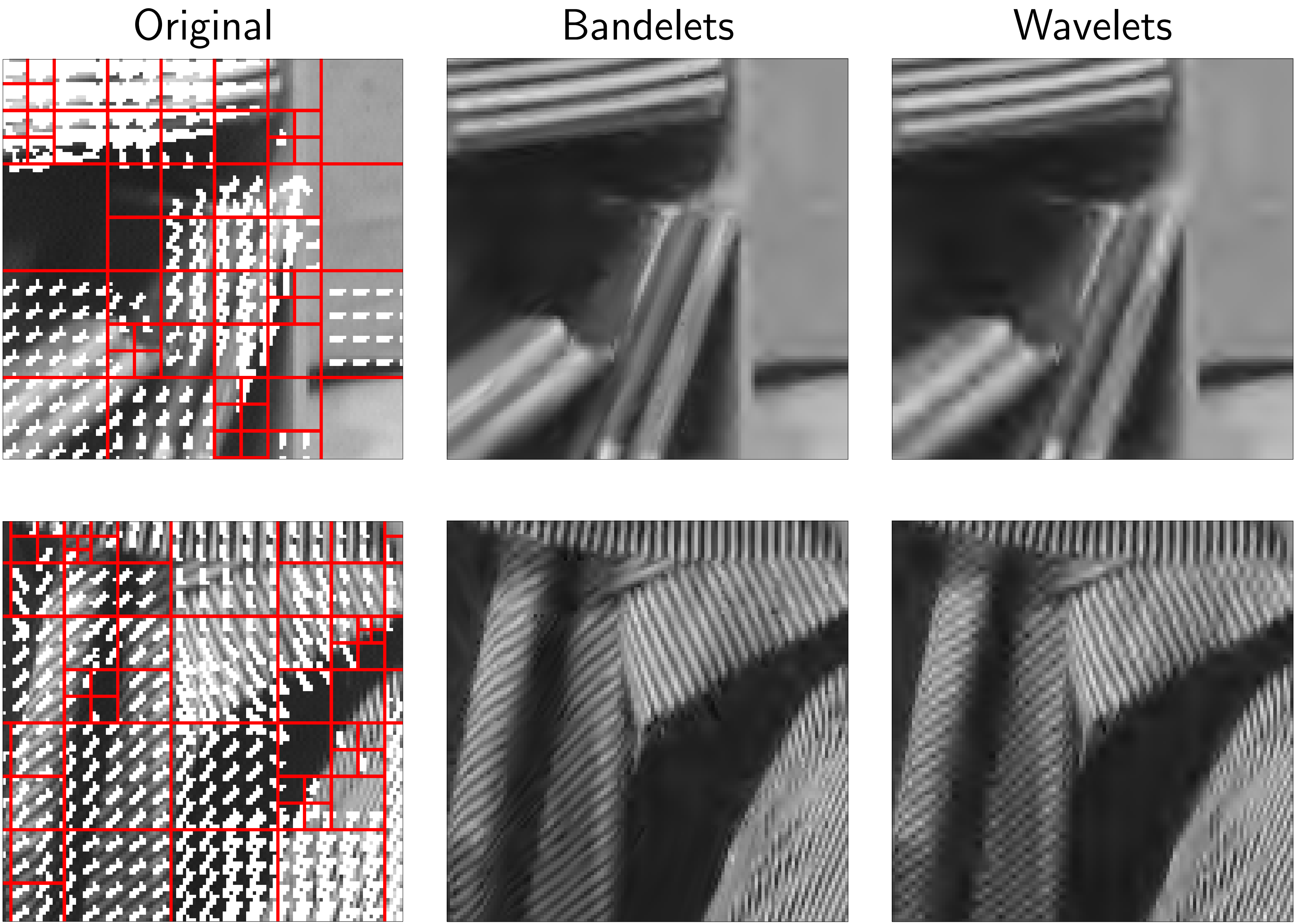


$R/N^2 = 0.40$ bpp

Bandelets (31.22 db)

Wavelets (29.68 db)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- Paper: `http://www.cmap.polytechnique.fr/~lepennec`