Sparse Geometrical Image Representation with Bandelets and Application to Deconvolution

Erwan Le Pennec and Stéphane Mallat

CMAP

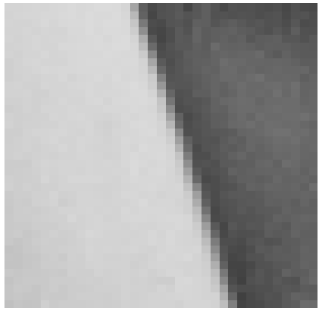
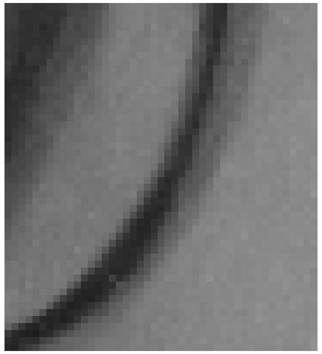
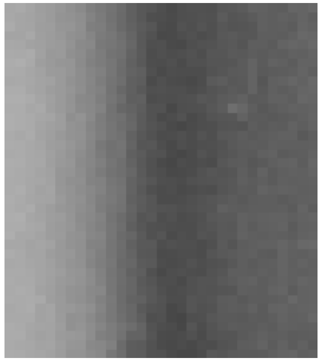
École Polytechnique

Geometrical Image Representation

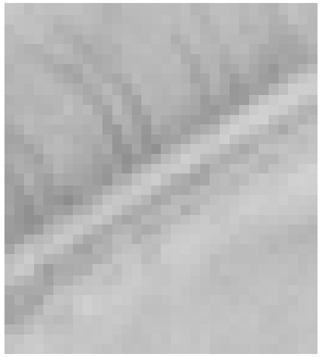
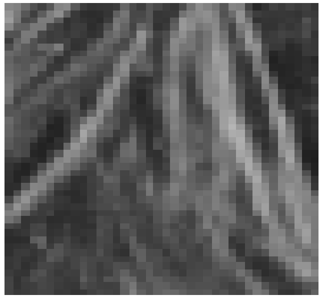
- Most signal processing applications requires to build sparse signal representations: compression, noise removal, restauration, pattern recognition...
- Need to take advantage of geometrical image regularity to improve representations.
- Second generation image coding dream : a bridge between *Image processing* and *Computer Vision*.
- Building harmonic analysis representations (wavelets) on manifolds (geometry).

Edge Detection: an Ill Posed Problem


• Edges are blured singularities.

• Where are the edges ?

• How can the estimation of geometry become well-posed ?

Overview

1. Sparse representations and wavelets

2. Description and detection of geometry

3. Orthogonal Bandelets adapted to the geometry

4. M-term image approximation theorem with bandelets

5. Application to deconvolution

Sparse Representation in a Basis

- A signal f is decomposed in an orthonormal basis $\mathcal{B} = \{g_m\}_{m \in \ N}$:

$$f = \sum_{m=0}^{+\infty} \langle f, g_m \rangle \, g_m \ ,$$

and approximated by M vectors chosen adaptively

$$f_M = \sum_{m \in I_M} \langle f, g_m \rangle \, g_m$$

to minimize

$$\|f - f_M\|^2 = \sum_{m \notin I_M} |\langle f, g_m \rangle|^2$$

- I_M should correspond to the M largest inner products :

$$I_M = \{m, |\langle f, g_m \rangle| > T_M\} : \text{thresholding}$$

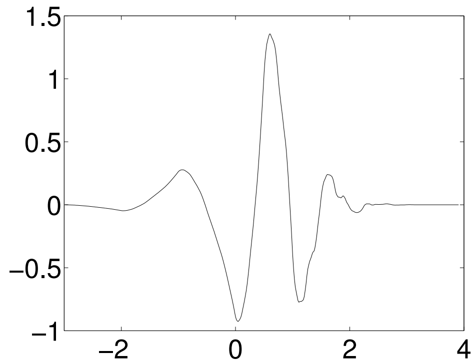
- **Problem :** How to choose the basis $\mathcal{B}$ so that

$$\|f - f_M\| \leq C M^{-\alpha} \ \text{ with } \alpha \text{ large ?}$$

1D Wavelet Basis of $\mathbf{L^2}[0,1]$

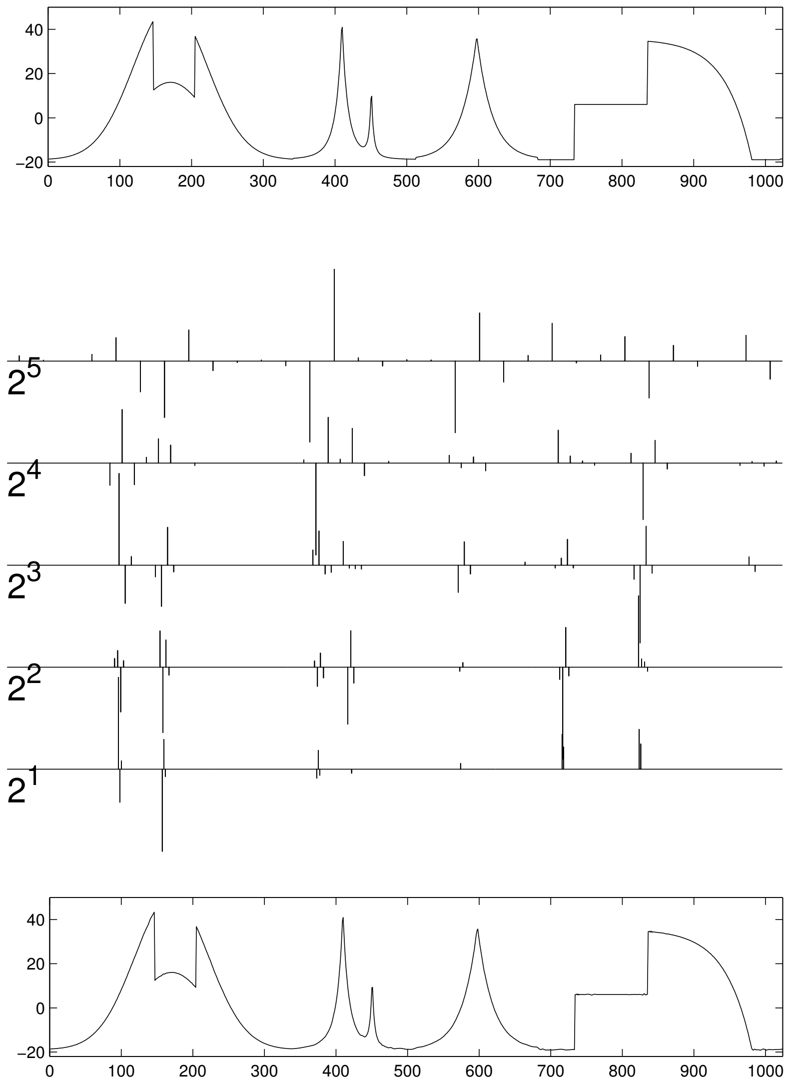
• Constructed with 1 mother wavelet $\psi(x)$ which is scaled by 2^j and translated by $2^j n$

$$\psi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\,\psi\left(\frac{x - 2^j n}{2^j}\right)\,.$$



• $\mathcal{B} = \left\{\psi_{j,n}\right\}_{j\in\ N\ ,\ 2^j n\in[0,1)}$ is an orthonormal basis of $\mathbf{L^2}[0,1]$.

Non-Linear Approximation in a Wavelet Basis

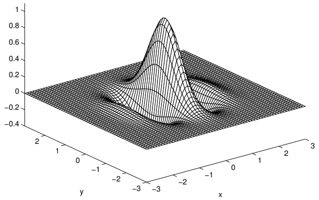
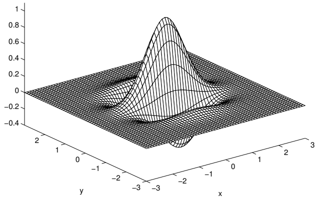
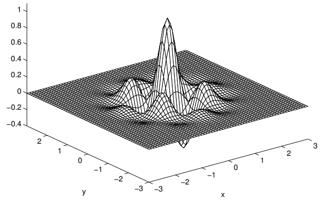


- $\|f - f_M\|^2 = O(M^{-2\alpha})$ where α is the uniform Lipschitz regularity between singularities.

2D Wavelet Basis of $\mathbf{L^2}[0,1]^2$

• Constructed with 3 wavelets $\psi^k(x_1, x_2)$ with $k = 1, 2, 3$ *(Meyer, M.)* which are scaled by 2^j and translated by $2^j(n_1, n_2)$

$$\psi^k_{j,n}(x_1, x_2) = \frac{1}{2^j}\, \psi^k\left(\frac{x_1 - 2^j n_1}{2^j}, \frac{x_2 - 2^j n_1}{2^j}\right) .$$

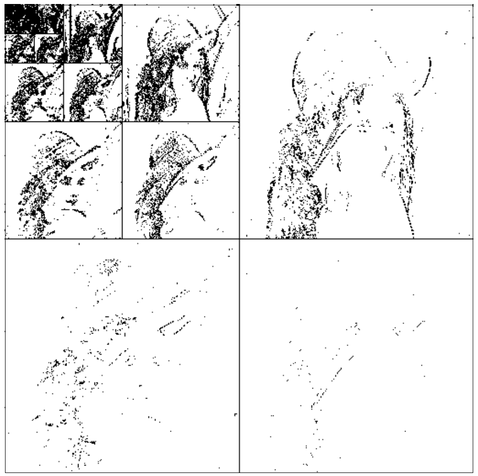
• $\mathcal{B} = \left\{\psi^k_{j,n}\right\}_{j\in\ N\ ,\, 2^j n \in [0,1)^2\ ,\, 1 \le k \le 3}$ is an orthonormal basis of $\mathbf{L^2}[0,1]^2$.

Successes and Failures of Wavelet Bases

- Representation: images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f

Significant coefficients



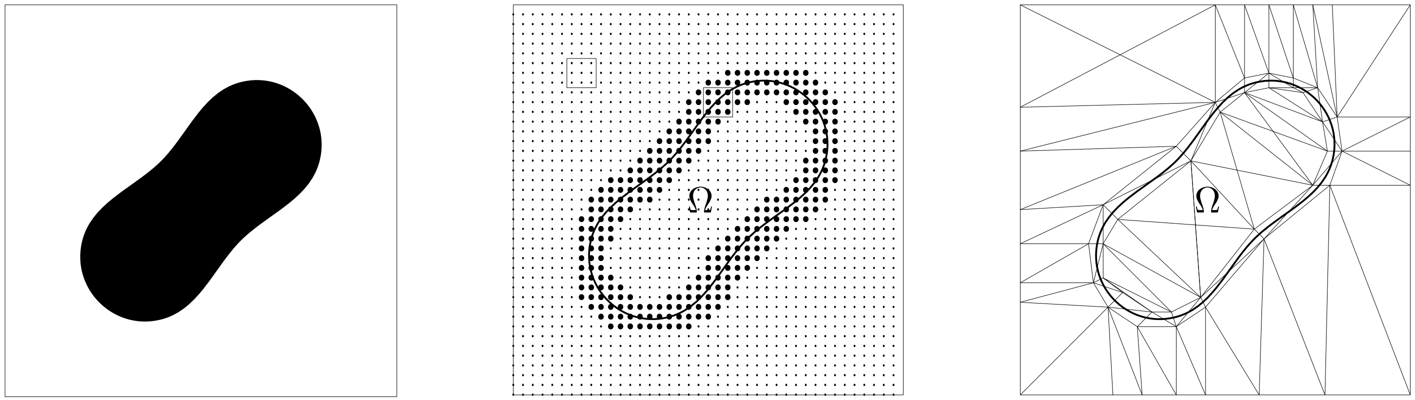
f_M

- *(Cohen, DeVore, Petrushev, Xue)*: Optimal for bounded variation functions: $\|f - f_M\|^2 \leq C\,\|f\|_{TV}\,M^{-1}$
- But: does not take advantage of any geometric regularity when it exists.

Taking Advantage of Geometrical Regularity

Most images have level sets that are regular geometrical curves.

Let $f = \mathbf{1}_\Omega$, where the boundary $\partial\Omega$ is regular: $\mathbf{C}^\alpha$ with $\alpha \geq 2$.



- With M wavelets: $\|f - f_M\|^2 \leq C\, M^{-1}$.
- Piece-wise linear with M triangles: $\|f - f_M\|^2 \leq C\, M^{-2}$.
- With M higher order geometric elements: $\|f - f_M\| \leq C\, M^{-\alpha}$.

- Curvelet bases (*Candes, Donoho*): $\|f - f_M\|^2 \leq C\,(\log M)\, M^{-2}$.
- Contourlet bases (*Minh-Do, Vetterli*).
- Edge adapted (*Cohen, Matei*): $\|f - f_M\|^2 \leq C\, M^{-2}$?

Blured and Noisy Geometry

Piecewise regular images $g(x)$ are blured and noisy:

$$f(x) = g \star \phi_s(x) + b(x) \quad \text{with} \quad \phi_s(x) = \frac{1}{s}\,\phi(\frac{x}{s}) \ .$$

- ϕ is unknown but $\mathbf{C}^\infty$ with a support in $[-1, 1]$.
- $s > 0$ is unknown and may vary with x.
- $b(x)$ is a "noise".

Problems:

- Represent and detect the geometry.
- Take advantage of the geometrical regularity.

Anisotropic 2D Wavelet Basis

- 1D wavelet basis of $\mathbf{L^2}[0,1]$:

$$\{\psi_{j,n}(x) = 2^{-j/2}\psi(2^{-j}(x - 2^j n))\}_{j\in\mathbb{Z}, 2^j n \in [0,1]} \ .$$

- *Anisotropic wavelet basis* of $\mathbf{L^2}[0,1]^2$:

$$\{\psi_{j_1,n_1}(x_1)\,\psi_{j_2,n_2}(x_2)\}_{j_1,n_1,j_2,n_2} \ .$$

- Let $g(x_1,x_2)$ be $\mathbf{C}^\alpha$ for $x_1 < a$ and $x_1 > a$ or for $x_2 < b$ and $x_2 > b$.
  If $f = g$ or $f = g \star \phi_s$ then its approximation f_M from M anisotropic wavelet satisfies

$$\|f - f_M\|^2 \le C\, M^{-\alpha} \ .$$

Horizontal and Vertical Geometric Flow

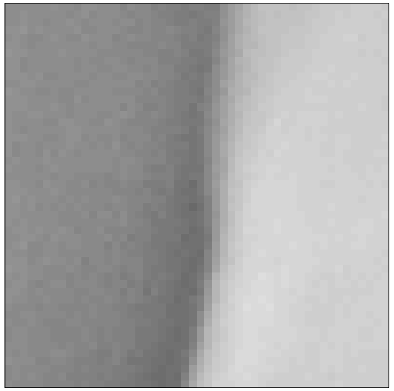
- Over a domain Ω the geometric flow is a parallel vector field $\vec{\tau}(x_1, x_2)$ with

$$\vec{\tau}(x_1, x_2) = \vec{\tau}(x_2) \quad \text{or} \quad \vec{\tau}(x_1, x_2) = \vec{\tau}(x_1)$$

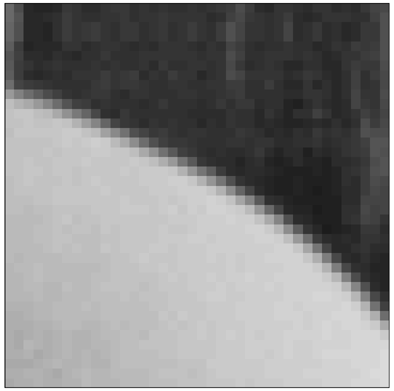
which minimizes

$$\int_\Omega |\vec{\nabla} f(x_1, x_2) . \vec{\tau}(x_1, x_2)|^2 dx_1 dx_2 = \int_\Omega |\frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)}|^2 dx_1 dx_2$$

$\vec{\tau}(x_1, x_2) = \vec{\tau}(x_2)$



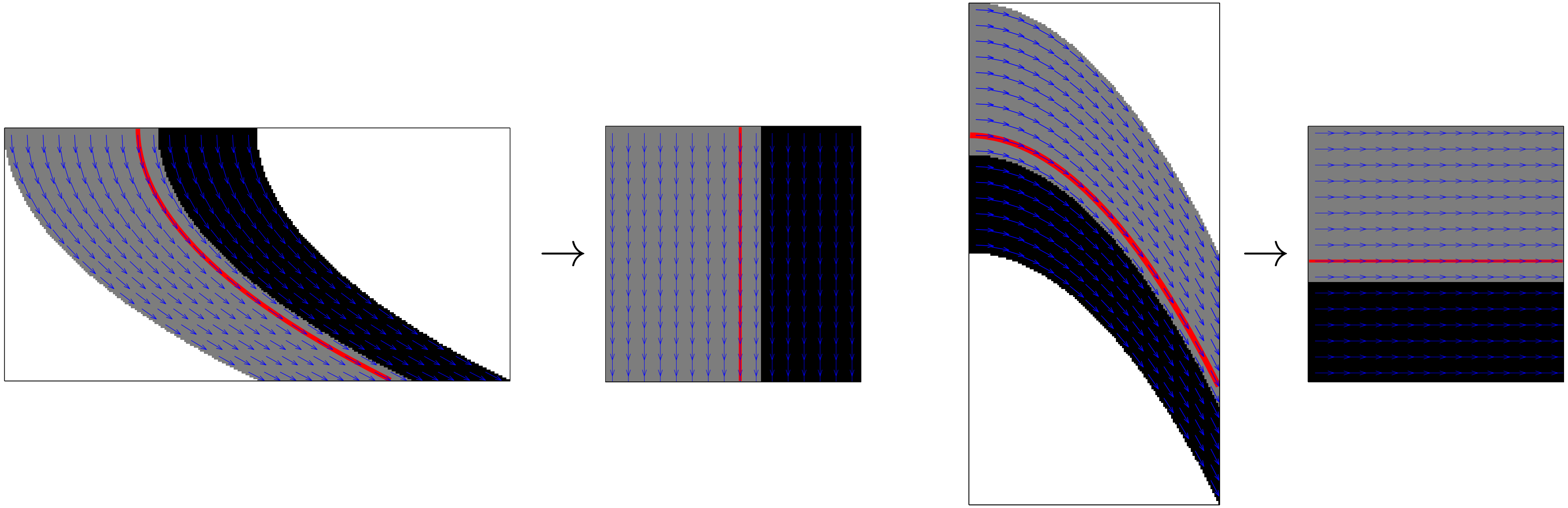
$\vec{\tau}(x_1, x_2) = \vec{\tau}(x_1)$



Flow, Integral Curve and Deformation

- Let $x_1 = c(x_2)$ be an integral curve of flow $\vec{\tau}(x_1, x_2) = \vec{\tau}(x_2)$. The image $f_c(x_1, x_2) = f(x_1 + c(x_2), x_2)$ has a vertical flow.

- If $\vec{\tau}(x_1, x_2) = \vec{\tau}(x_1)$ then for an integral curve $x_2 = c(x_1)$, the image $f_c(x_1, x_2) = f(x_1, x_2 + c(x_1))$ has a horizontal flow.



Bandelet Basis

Decomposing $f(x_1 + c(x_2), x_2)$ in an orthonormal anisotropic wavelet basis

$$\{\psi_{j_1,n_1}(x_1)\, \psi_{j_2,n_2}(x_2)\}_{j_1,j_2,n_1,n_2}$$

is equivalent to decompose $f(x_1, x_2)$ in the orthonormal *bandelet* basis

$$\{\psi_{j_1,n_1}(x_1 - c(x_2))\, \psi_{j_2,n_2}(x_2)\}_{j_1,j_2,n_1,n_2} \ .$$

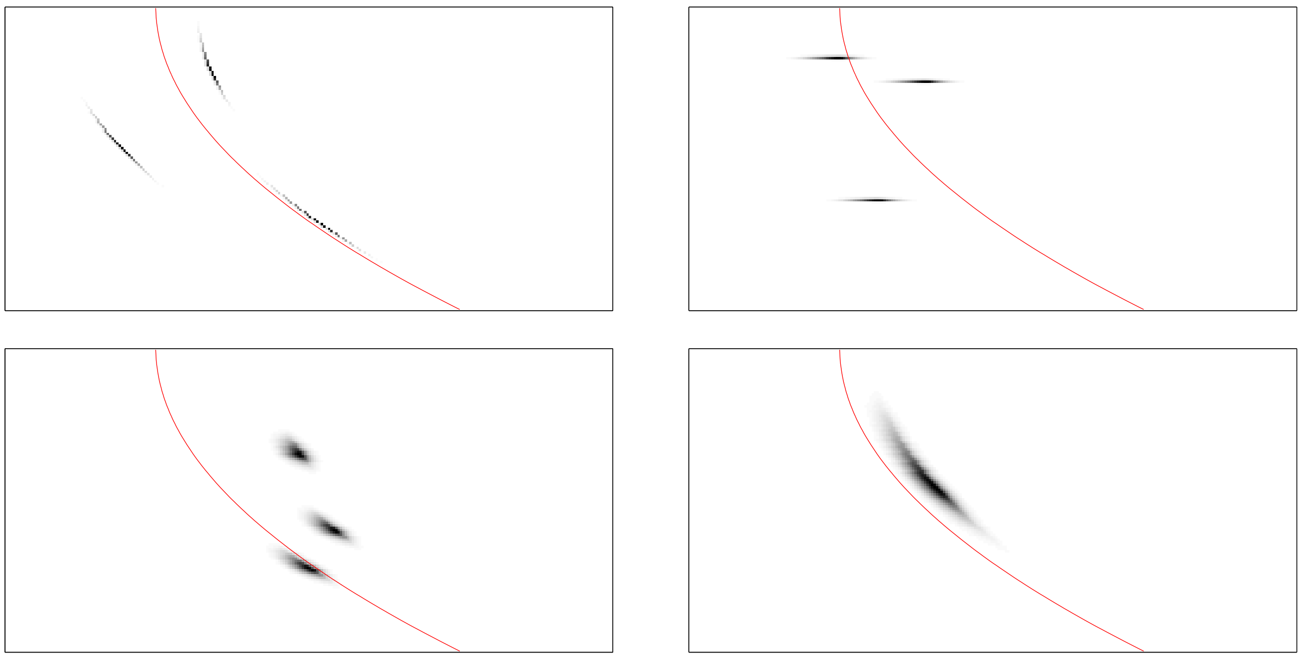


Image Geometry

- Image segmented in disjoint regions Ω_i with a geometric flow $\vec{\tau}_i(x)$.
- Complement $C = [0,1]^2 - \cup_i \Omega_i$ (no preferential direction).

The image geometry is specified by:

- The boundary $\partial\Omega_i$ of each Ω_i.
- An integral curve $c_i(x)$ of the flow $\vec{\tau}_i(x)$ in each Ω_i.

Bandelet in Segmented Images

Composed of:

- Bandelets $\mathcal{B}_{\Omega_i}$ with supports in Ω_i.
- Isotropic wavelets $\mathcal{B}_w$ with supports in $C = [0,1]^2 - \cup_i \Omega_i$.
- Border bandelets $\mathcal{B}_{\partial\Omega_i}$ with supports intersecting the border $\partial\Omega_i$.

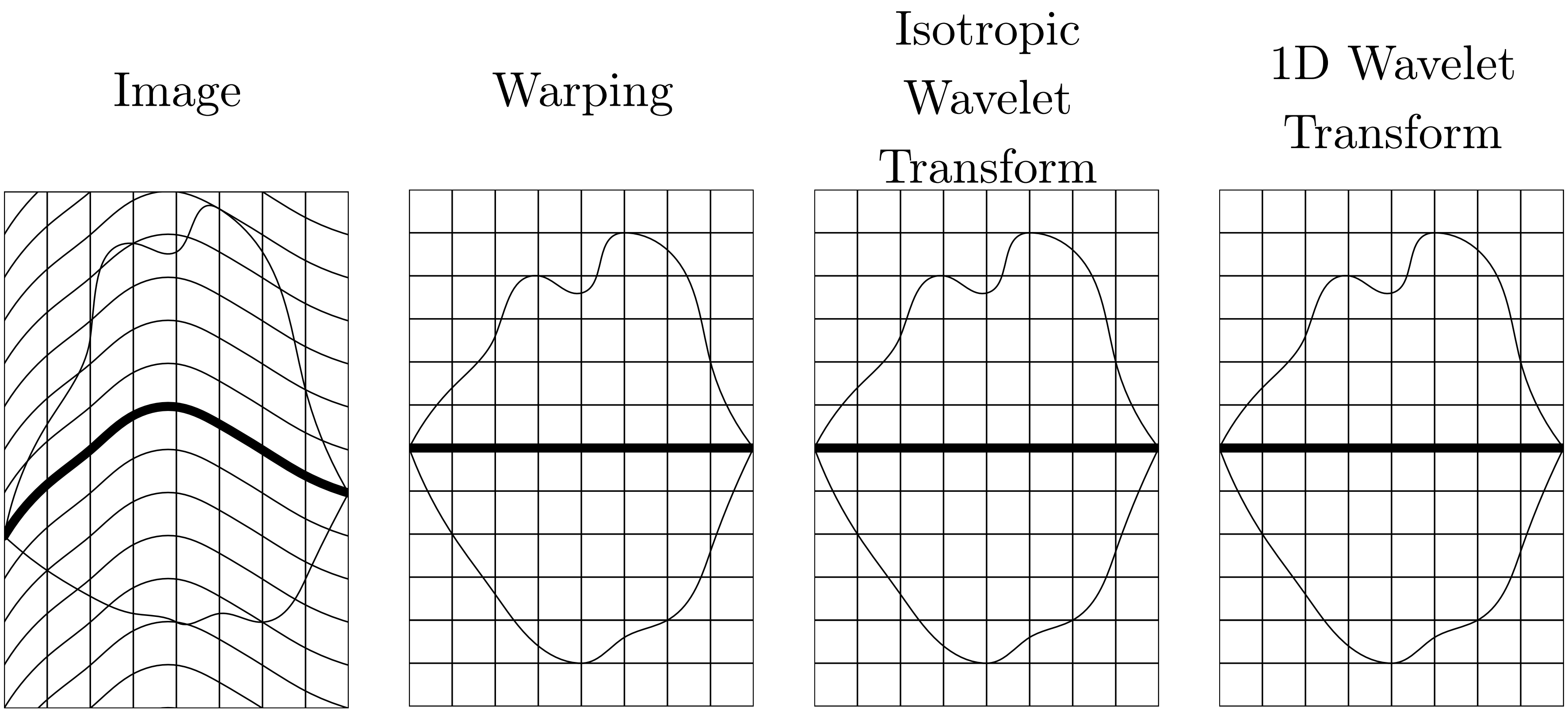
so that the resulting family

$$\mathcal{B}_b = (\cup_i \mathcal{B}_{\Omega_i}) \cup \mathcal{B}_w \cup (\cup_i \mathcal{B}_{\partial\Omega_i})$$

is a frame or a basis of $\mathbf{L^2}[0.1]^2$.

Bandeletization

- Fast transform ($O(N)$)



- The warping requires an interpolation operator.

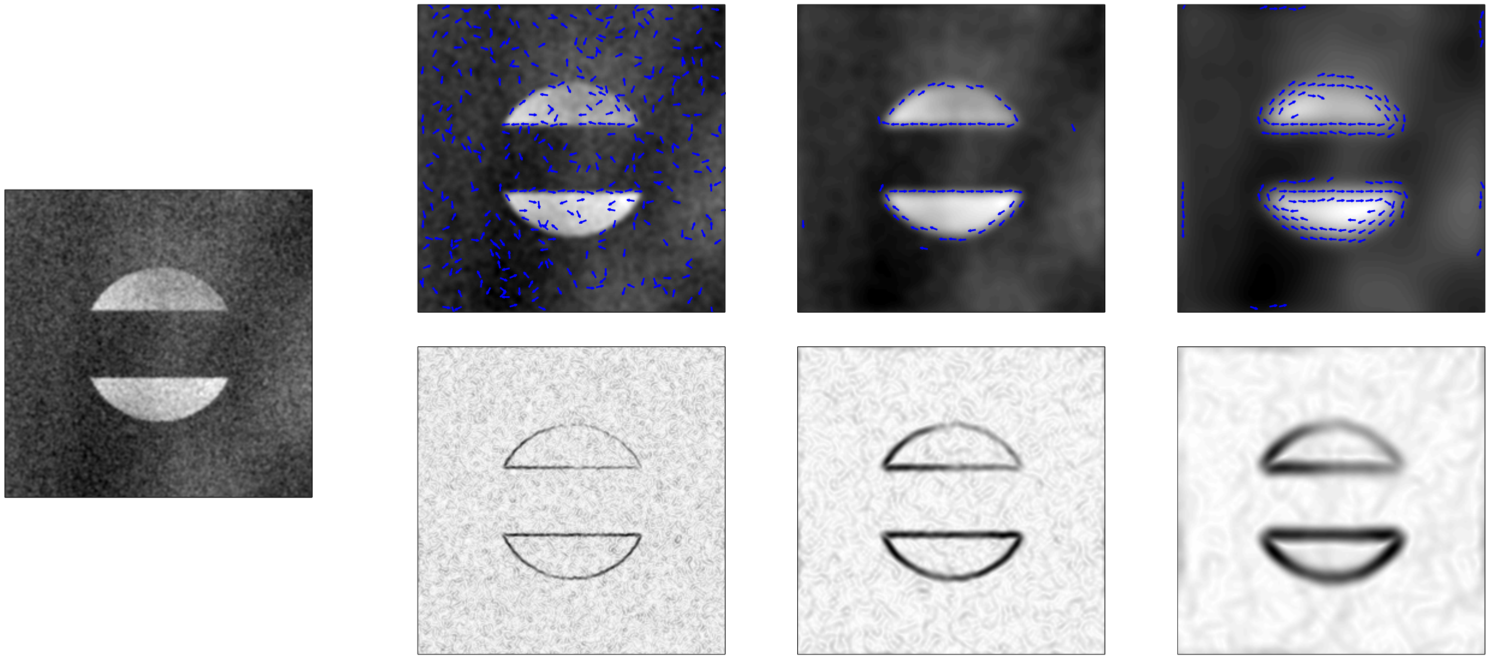
Multiscale Estimation of Geometry

- Blured and noisy image in $x = (x_1, x_2)$:

$$f(x) = g \star \phi_s(x) + b(x) \ .$$

- $b(x)$: White Noise ($\mathcal{N}(0, \sigma^2)$).
- Dilated kernel $h_{2^j}(x) = h(2^{-j}x)$ where $h(x)$ is isotropic and compactly supported:

$$\vec{\nabla}(f \star h_{2^j})(x) = \vec{\nabla}(g \star h_{2^j} \star \phi_s)(x) + W \star \vec{\nabla} h_{2^j}(x) \ .$$

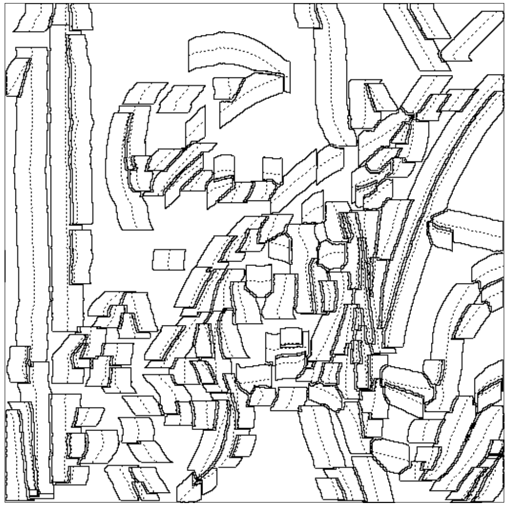


Approximation of Geometry

- Approximation of the bandelet basis.
- Positions of the begining and ending points of the curve c_i.
- Non-linear approximation of $c_i(x)$ in a 1D wavelet basis, with an adapted threshold Δ_i:

$$c_i = \sum_{j,n} \langle c_i\,,\, \psi_{j,n} \rangle\, \psi_{j,n} \quad \Rightarrow \quad \tilde{c}_i = \sum_{|\langle c_i\,,\, \psi_{j,n}\rangle| > \Delta_i} \langle c_i\,,\, \psi_{j,n} \rangle\, \psi_{j,n}\ .$$

c_i and Ω_i $\Rightarrow$ $\tilde{c}_i$ and $\tilde{\Omega}_i$



M-Term Approximation

- Two kinds of coefficents : geometry and decomposition.
- Geometry = choice of the bandelet basis $\mathcal{B}$ with adjusted thresholds Δ_i:
  - 2 extremity points for each c_i.
  - $M_{g,i}$: number of 1D wavelet coefficients of c_i larger than Δ_i.

  c_i, Ω_i are thus approximated by $\widetilde{c}_i, \widetilde{\Omega}_i$.
- Decomposition = coefficients of f in the chosen bandelet basis $\mathcal{B}$ larger than Δ:
  - $M_{b,i}$: for bandelets and border bandelets corresponding to $\tilde{\Omega}_i$.
  - $M_{w,C}$: for wavelets in the complement.
- Resulting M-term approximation: f_M with

$$M = \sum_i \left(2 + M_{g,i} + M_{b,i}\right) + M_{w,C} \ ,$$

Optimization of the Geometry

- Choice of the basis that leads to the most sparse representation.
- Error $\|f - f_M\|^2$ depends mostly on Δ.
- To minimize $M = \sum_i \left(2 + M_{g,i} + M_{b,i}\right) + M_{w,C}$, for each Ω_i find Δ_i that minimizes $M_{b,i} + M_{g,i}$.

Approximation of Piecewise Regular Images

Theorem: Suppose that g is $\mathbf{C}^\alpha$ in $[0,1]^2 - \{e_i\}_{1\leq i\leq I}$ and the e_i are $\mathbf{C}^\alpha$ curves.
If $f = g$ or $f = g \star \phi_s$ for any $s > 0$ then

$$\|f - f_M\|^2 \leq C\, M^{-\alpha}\ .$$

- Unknown degree of smoothness α.
- Unknown smoothing kernel ϕ_s.
- Optimal over Donoho Star Shape class.
- Improvement over 2D wavelets for which $\|f - f_M\|^2 \leq C\, M^{-1}$.
- Improvement over curvelets for which $\|f - f_M\|^2 \leq C\,(\log M)\, M^{-2}$.
- Estimation of the geometry.

Noise Removal with Thresholding

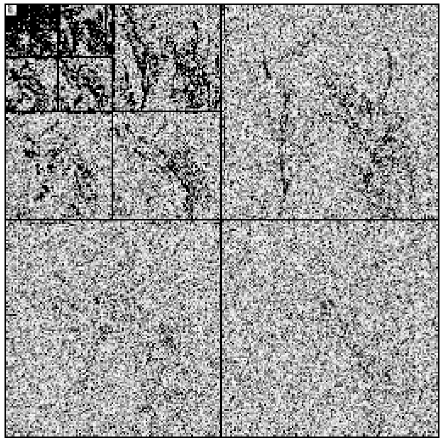
(Donoho, Johnstone)

f

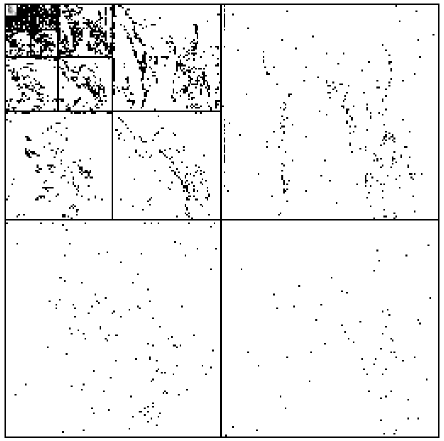
$X = f + W$

$F = X \star h$

$\langle X, \psi_{j,n}^k \rangle$



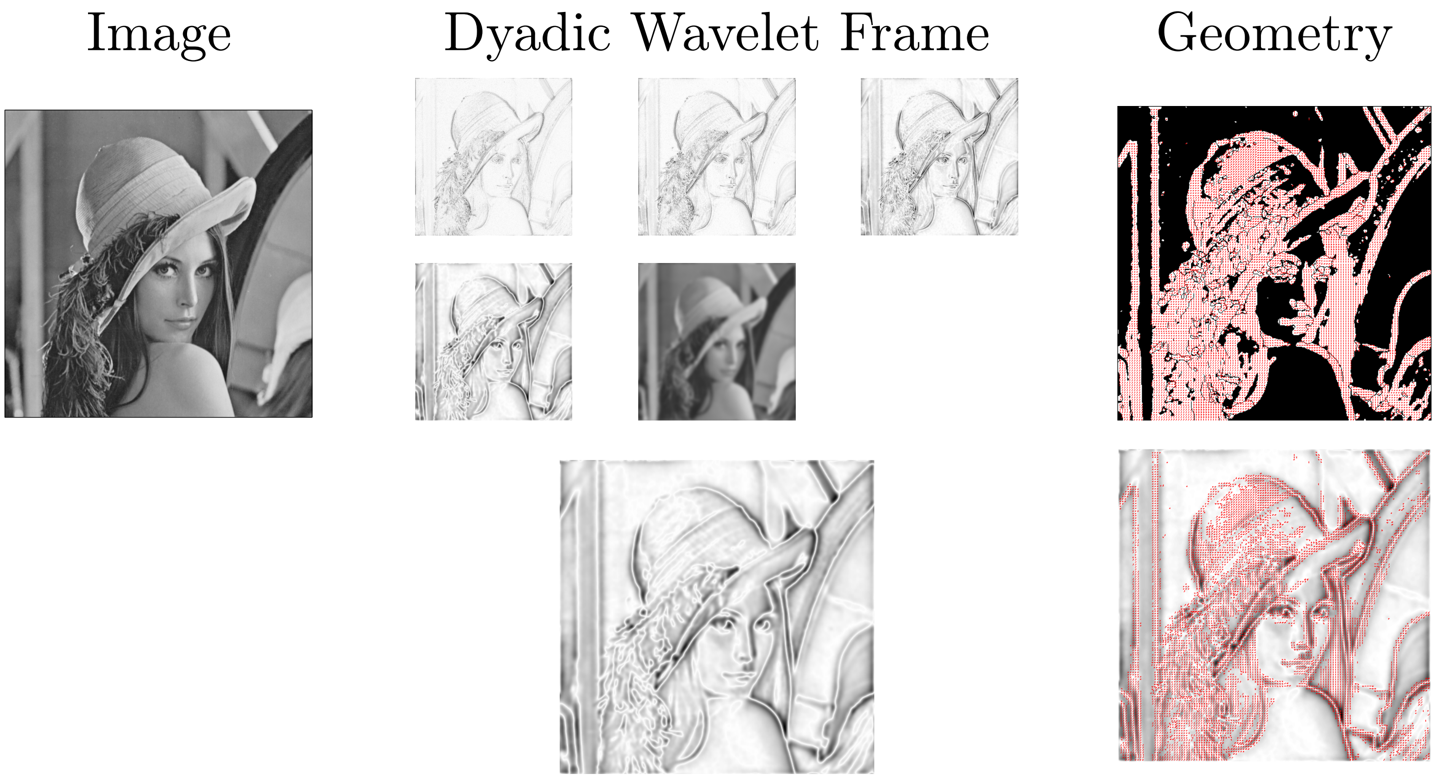
$\mathrm{Thresh}(\langle X, \psi_{j,n}^k \rangle)$



$F = T_{\mathcal{B}} X$

A frame of bandelets

- Translatation invariance.
- Dyadic Wavelet Frame + 1D Wavelet Transform along the flow (warping operator)



- Geometry estimation remains the same.

Deconvolution

- The signal observed is

$$Y = f \star u + W$$

where u is a known low-pass filter and W a white noise of variance σ^2.

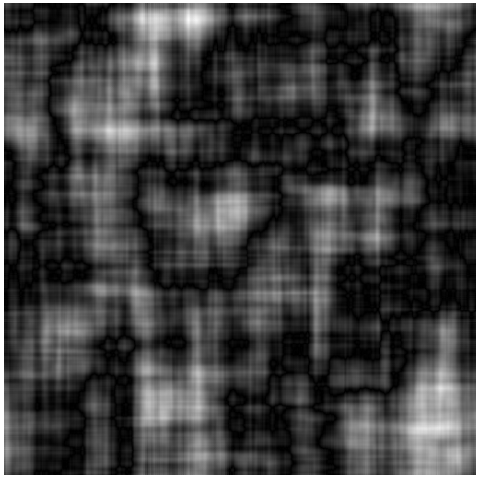
- Direct deconvolution

$$X = Y \star u^{-1} = f + W \star u^{-1}$$

Y

$\xrightarrow{u^{-1}}$

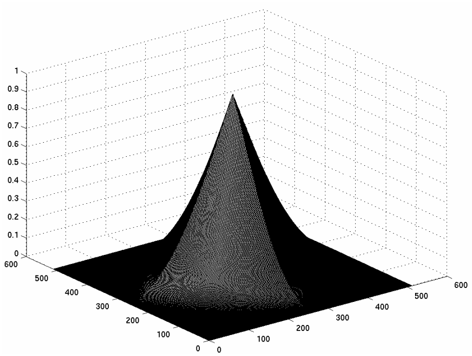
X

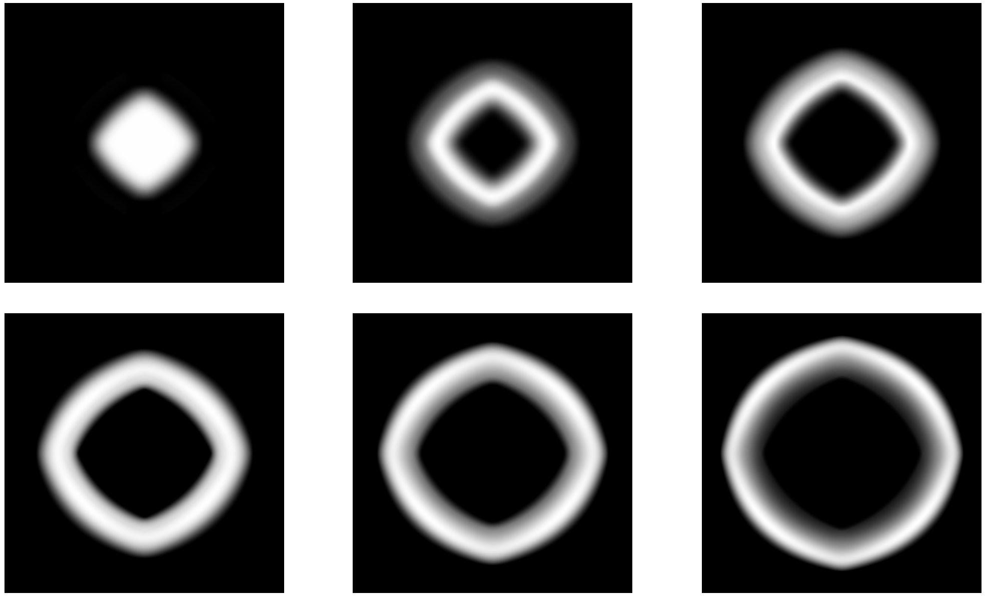


- Problem : The noise is no more white.

A frame of bandelets adapted to the deconvolution

- Dyadic wavelet frame replaced by an adapted frame.
- The low-pass filter should be almost diagonal in the frame.





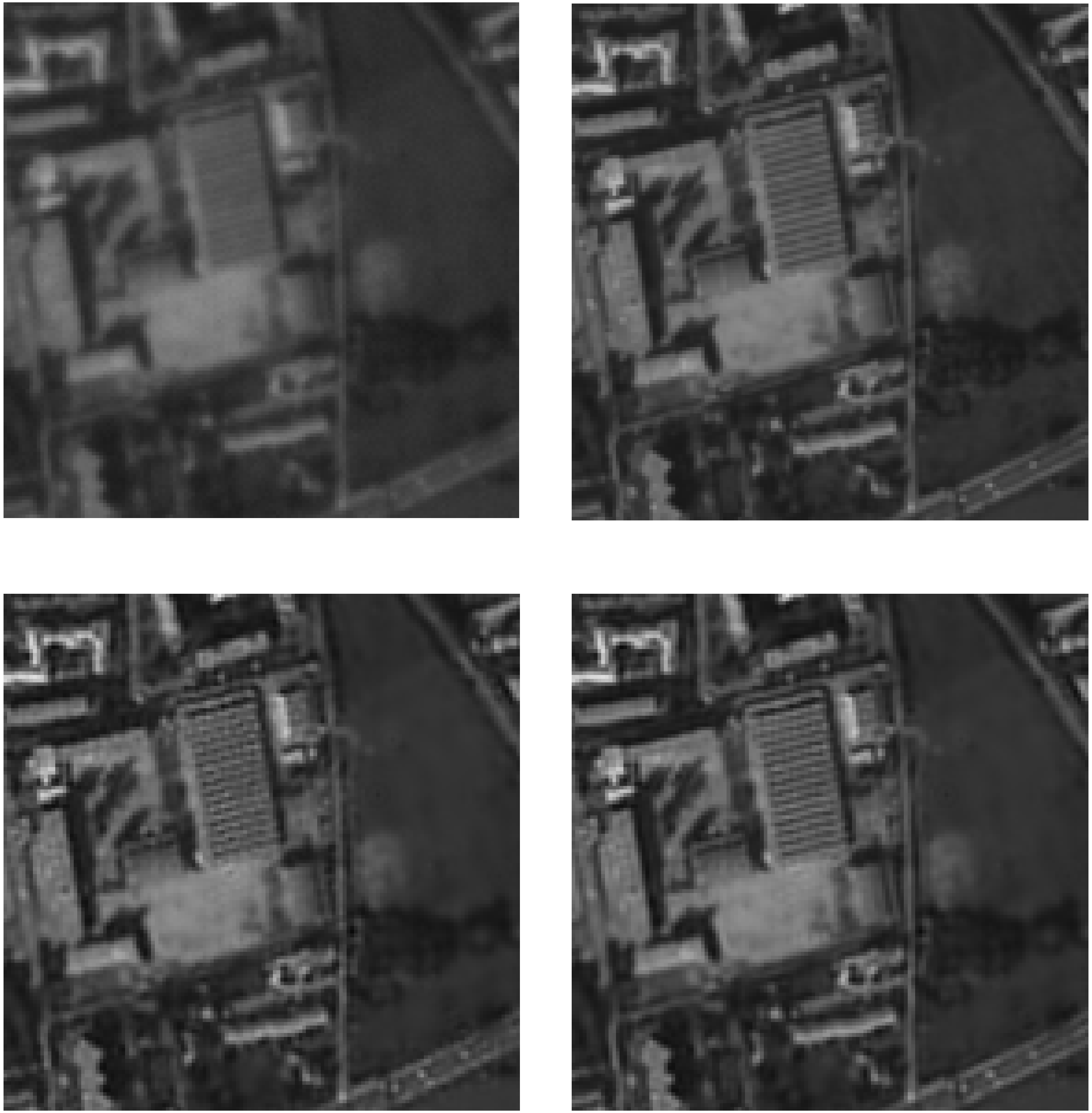
- Fast Transform :

Image

Adapted Frame

Geometry

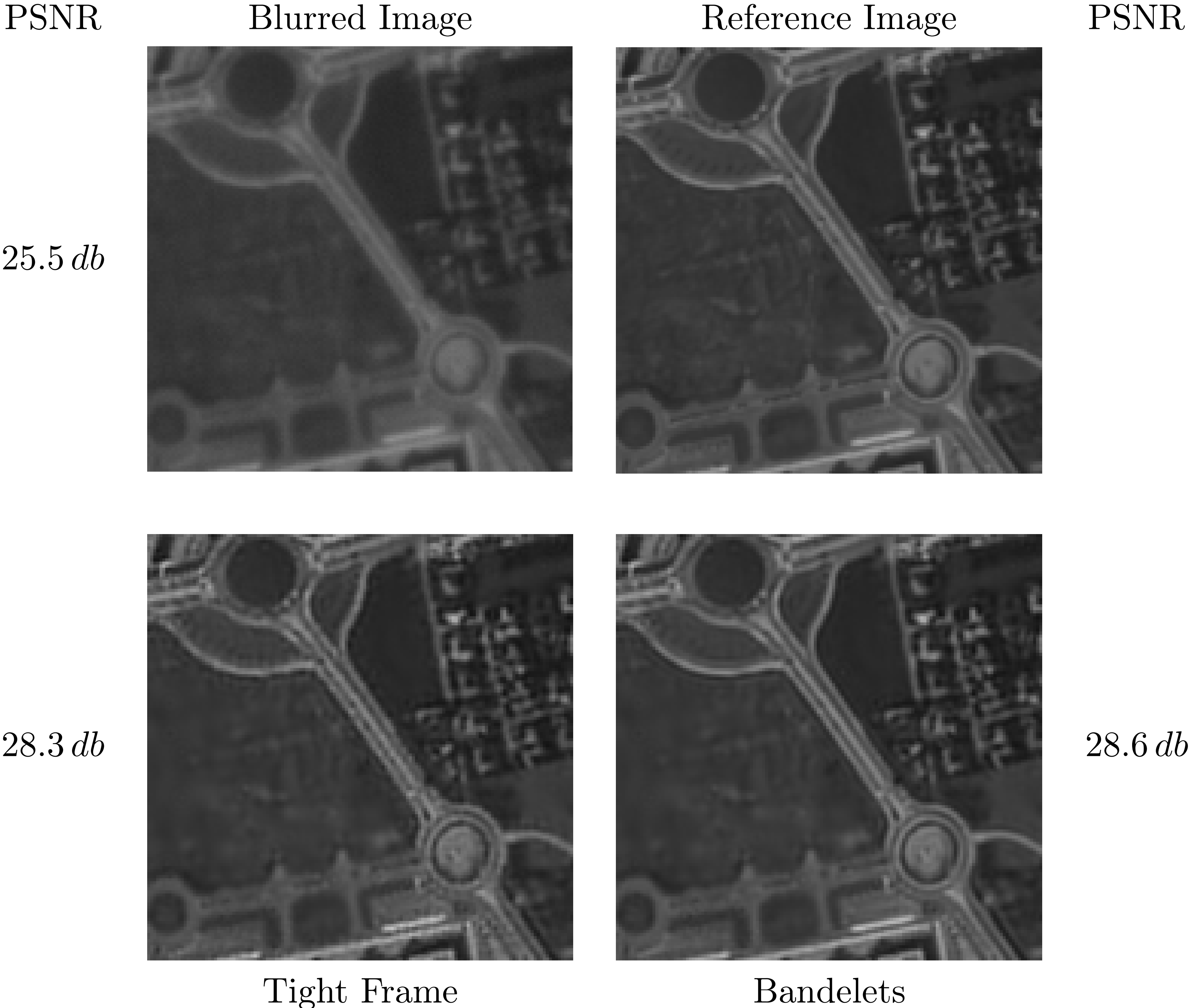
PSNR | Blurred Image | Reference Image | PSNR



25.5 db

28.3 db | 28.6 db

Tight Frame | Bandelets


PSNR
Blurred Image
Reference Image
PSNR
25.5 db
28.3 db
28.6 db
Tight Frame
Bandelets

Futur Improvements

- Design of the frame.
- Interpolation operator of the warping.
- Geometry estimation.
- Artefact removal.

Conclusion

- Bandelets provide sparse image representations in bases adapted to the image geometry.
- Applications to most image processing :
  - Still image coding.
  - Denoising and restoration by thresholding.
  - Video coding with regions.
  - Adapted to pattern recognition
- Mathematical issues :
  - Statistical consistency of the geometrical estimation.
  - Approximations theorems over adapted functionnal spaces.
  - Extension to d-dimensional spaces with $d > 2$.

Acknowledgment

- Jérôme Kalifa
- CNES
- Alcatel Space