Sparse Geometrical Image Representations with Bandelets

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Geometrical Image Representation

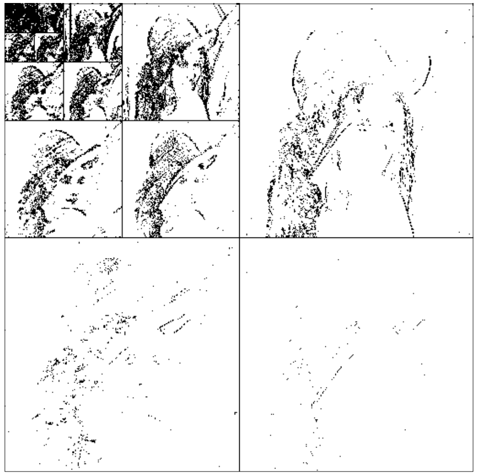
- Most signal processing applications requires to build sparse signal representations: compression, noise removal, restauration, pattern recognition...
- Need to take advantage of geometrical image regularity to improve representations.
- Requires to relate harmonic analysis representations (wavelets), and geometrical structures.
- A bridge between image processing and computer vision: the second generation image code dream.

Successes and Failures of Wavelet Bases

Representation: images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f

Significant coefficients



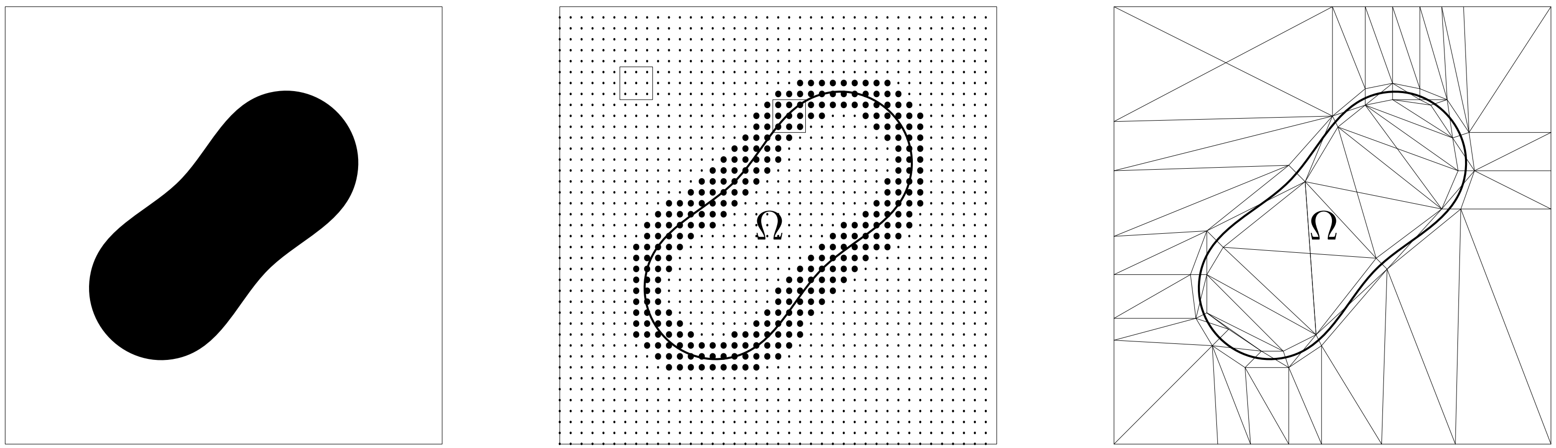
f_M

- *(Cohen, DeVore, Petrushev, Xue)*: Optimal for bounded variation functions: $\|f - f_M\|^2 = O(M^{-1})$
- But: does not take advantage of any geometric regularity when it exists.

Taking Advantage of Geometrical Regularity

Most images have level sets that are regular geometrical curves.

Let $f = \mathbf{1}_\Omega$, where the boundary $\partial\Omega$ is regular: $\mathbf{C}^s$ with $s \geq 2$.



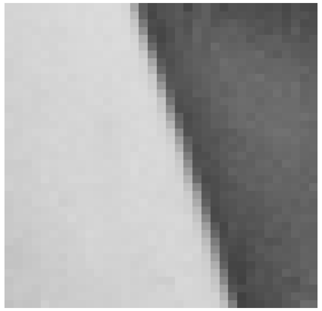
- With M wavelets: $\|f - f_M\|^2 = O(M^{-1})$.
- Piece-wise linear approximation with M triangles:

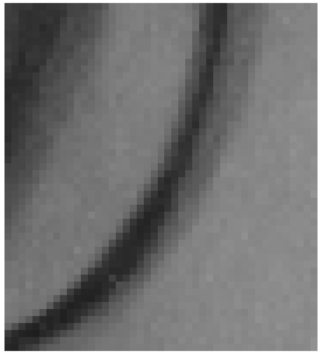
$\|f - f_M\|^2 = O(M^{-2})$.

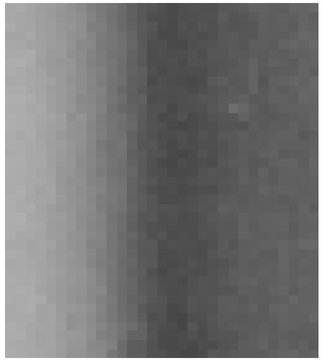
- With M higher order geometric elements: $\|f - f_M\|^2 = O(M^{-s})$.

Edge Detection: an Ill Posed Problem

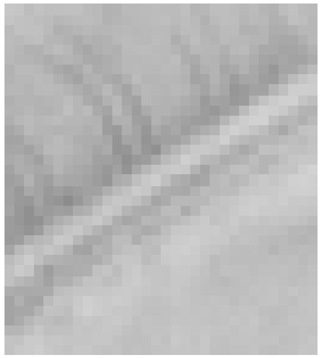
• Edges are not singularities.

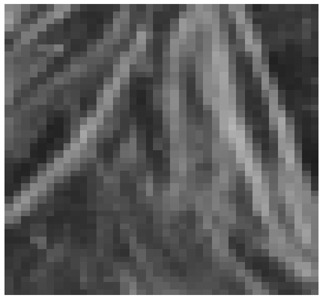






• Where are the edges ?





• Problem: find a stable process to extract and represent the geometry.

Overview

- Geometric Flow and Bandelets bases

- Image approximations with bandelets.

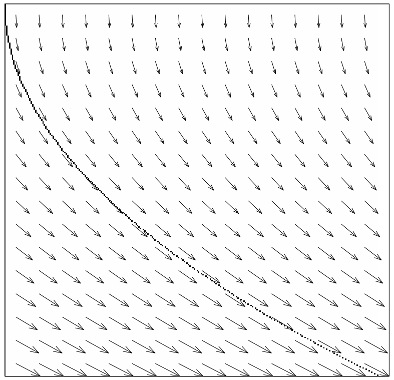
Geometric Flow

- A geometric flow is a vector field $\vec{\tau}(x_1, x_2)$ along which the image $f(x_1, x_2)$ has regular variations:

Simplifying the flow

• Geometric flow of f parametrized along x_2 and constant along x_1.

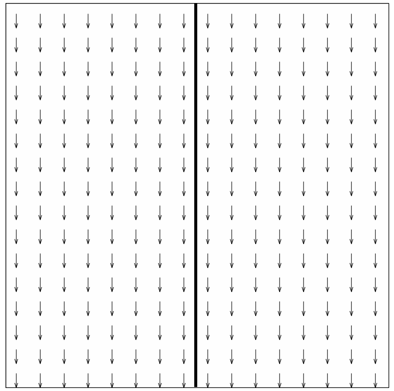
$$\vec{\tau}_f(x_1, x_2) = \Big(c'(x_1, x_2)\,,\,1\Big) = \Big(c'(x_2)\,,\,1\Big)\ .$$



• Integral curve c : points $(u_1 + c(x_2)\,,\,u_2 + x_2)$ with $c(x) = \int_{u_2}^{x} c'(t)\,dt$.

• The translated image $g(x_1, x_2) = f(x_1 + c(x_2), x_2)$ has a vertical flow.

Hyperbolic Wavelet Basis



- 1D wavelet basis of $\mathbf{L^2}[0,1]$:

$\{\psi_{j,n}(x) = 2^{-j/2}\psi(2^{-j}(x - 2^j n))\}_{(j,n)\in\mathbb{Z}^2}$.

- *Hyperbolic wavelet basis* of $\mathbf{L^2}[0,1]^2$:

$$\{\psi_{j_1,n_1}(x_1)\,\psi_{j_2,n_2}(x_2)\}_{(j_1,j_2)\in\mathbb{Z}^2\ (2^{j_1}n_1,2^{j_2}n_2)\in[0,1]^2}\ .$$

(Temlyakov): If f is regular along x_1 and/or along x_2

$$\left\| \frac{\partial^{r_1}\partial^{r_2} f(x_1,x_2)}{\partial x_1^{r_1}\partial x_2^{r_2}} \right\| < +\infty$$

then the non-linear approximation from M hyperbolic wavelet satisfies

$$\|f - f_M\| = O(M^{-\min(r_1,r_2)})\ .$$

Bandelet Basis

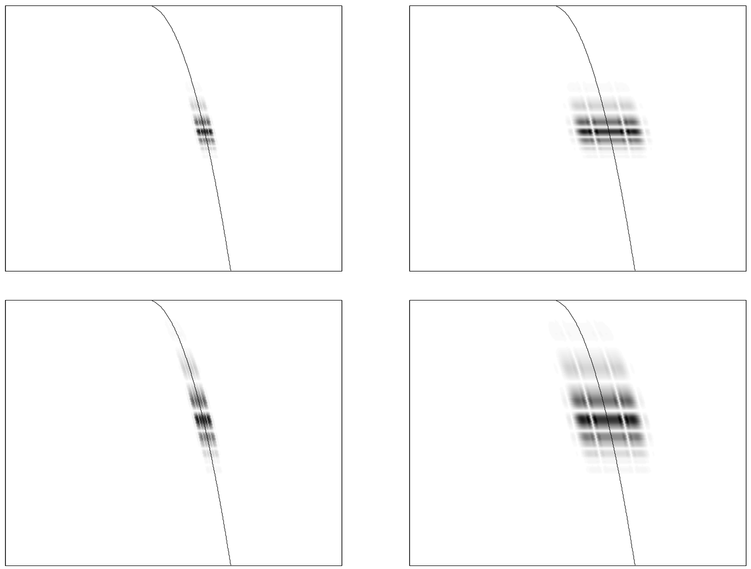
Decomposing $f(x_1 + c(x_2), x_2)$ in

$$\{\psi_{j_1,n_1}(x_1)\,\psi_{j_2,n_2}(x_2)\}_{(j_1,j_2)\in\mathbb{Z}^2\;(2^{j_1}n_1,2^{j_2}n_2)\in[0,1]^2}$$

is equivalent to decompose $f(x_1, x_2)$ in the *bandelet* basis:

$$\{\psi_{j_1,n_1}(x_1 - c(x_2))\,\psi_{j_2,n_2}(x_2)\}_{(j_1,j_2)\in\mathbb{Z}^2\;(2^{j_1}n_1,2^{j_2}n_2)\in[0,1]^2}\;.$$

Examples of bandelets:



Bandelet over a tube Ω

• Ω is a tube of length l and width d around a curve c starting at (x_1, x_2).

• Arbitrary extension of the flow c' outside Ω and associated bandelet basis

• Select the set B_Ω of (j_1, n_1, j_2, n_2) such that the support of $\psi_{j_1,n_1}(x_1 - c(x_2))\,\psi_{j_2,n_2}(x_2)$ is included in Ω.

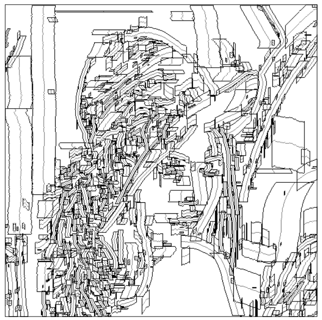
The resulting family of bandelets

$$\{\psi_{j_1,n_1}(x_1 - c(x_2))\,\psi_{j_2,n_2}(x_2)\}_{(j_1,n_1,j_2,n_2)\in B_\Omega}$$

is an orthogonal family of smooth functions having a support in Ω. It defines a basis of a space $\mathbf{V}_\Omega$.

Segmentation of the Image Plane

- The image is segmented in tubes Ω_i where the geometric flow is parallel, characterized by an integral curve c_i, and zones where there is no geometical regularity.
- A bandelet family is constructed over each tube.



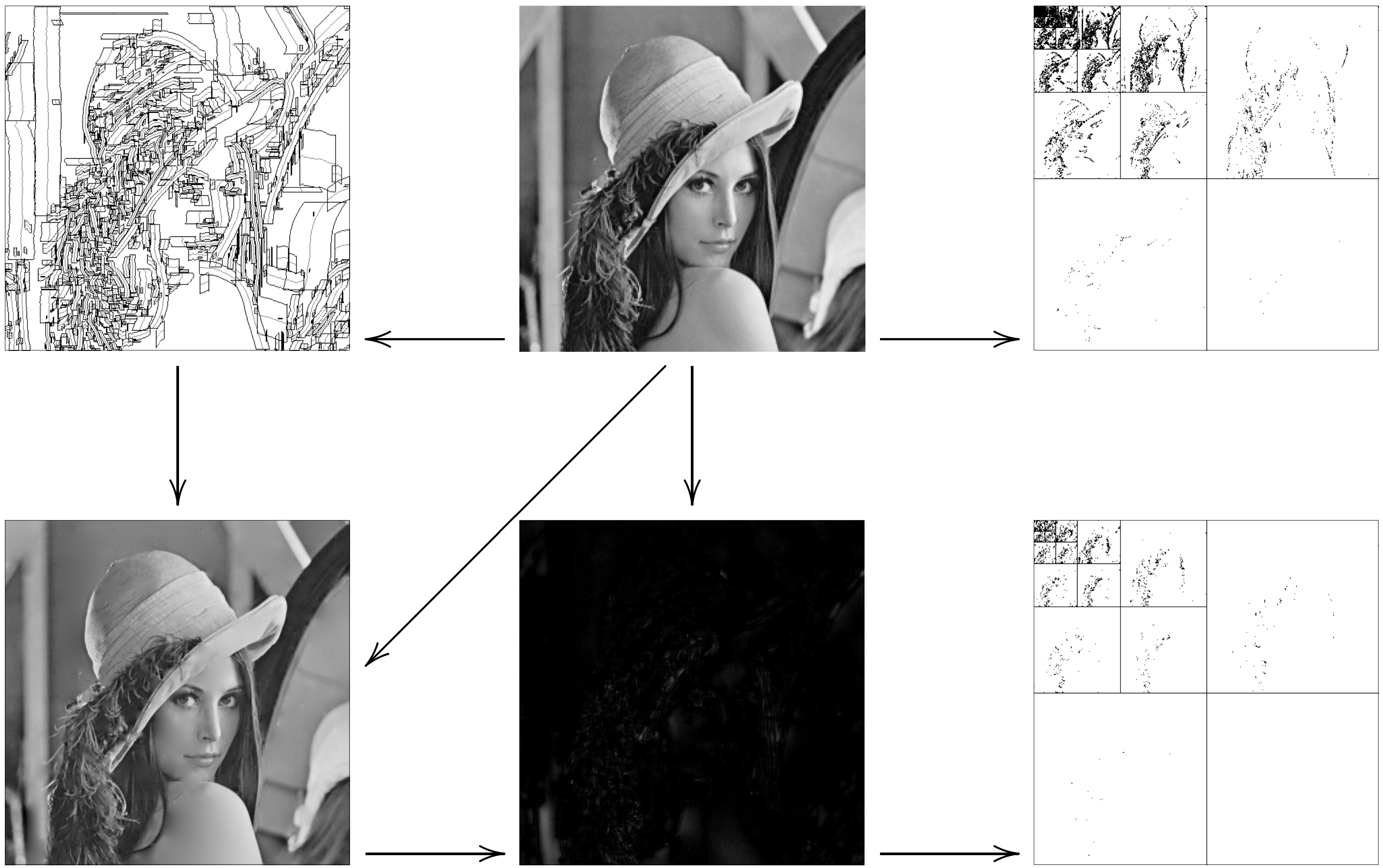
Let $\mathbf{V} = \oplus_i \mathbf{V}_{\Omega_i}$. The residue

$$r = f - P_{\mathbf{V}} f$$

is decomposed in a 2D wavelet basis of $\mathbf{L^2}[0,1]^2$

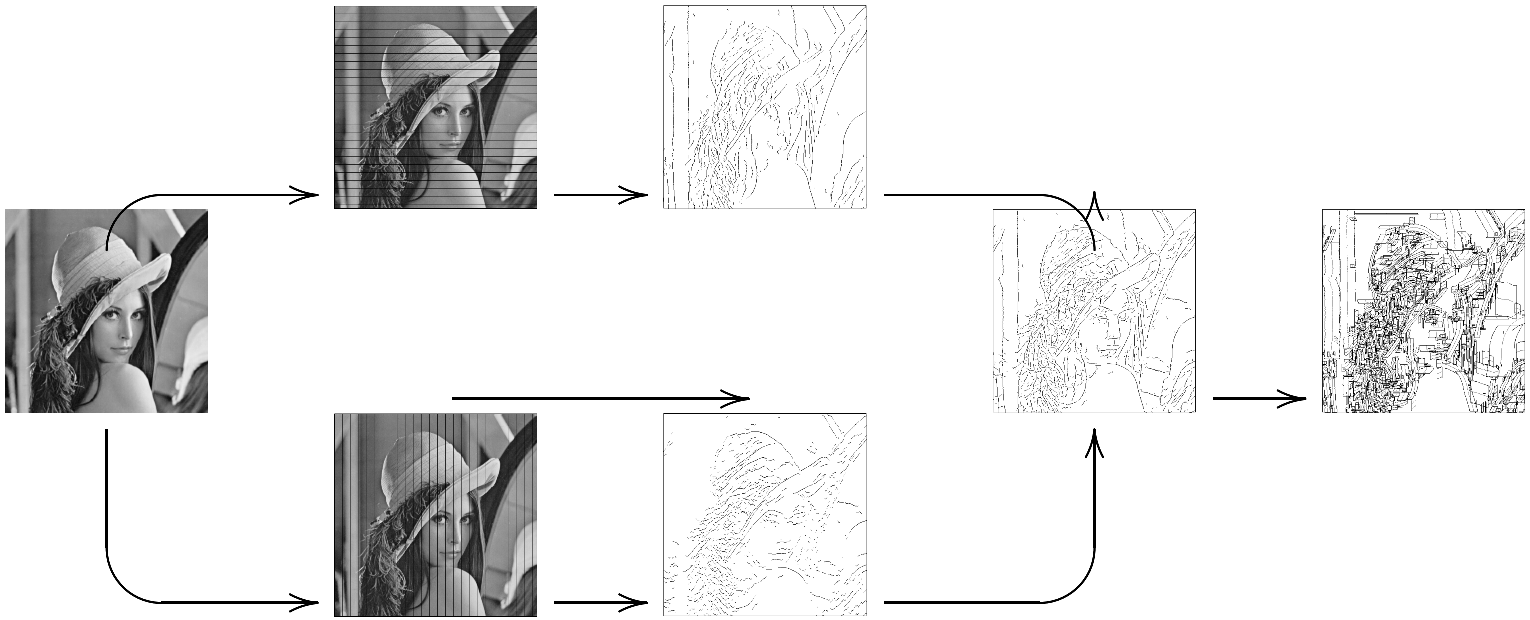
$$\left\{ \psi_{j,n}^k \right\}_{j \in N, 2^j n \in [0,1)^2, 1 \leq k \leq 3} .$$

Lena Bandelet Decomposition



Detection of Tubes

- Detection of significant 1D energy points along lines and columns.
- Chaining by minimizing the variation of the 1D wavelets coefficients.
- Rectangular region growing with limited intersection.

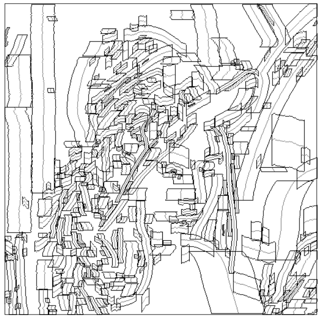


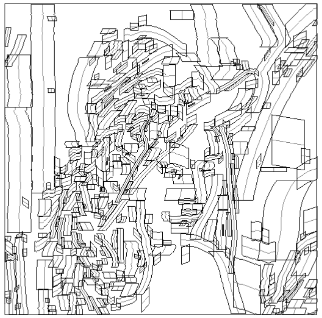
Approximation of Geometry

Each tube Ω_i and its flow $c'_i(x)$ is defined by an integral curve $c_i(x)$. It is approximated by $\tilde{c}_i$ in a wavelet basis:

$$c_i = \sum_{j,n} \langle c_i \, , \, \psi_{j,n} \rangle \, \psi_{j,n} \quad \Rightarrow \quad \tilde{c}_i = \sum_{|\langle c_i \, , \, \psi_{j,n} \rangle| > \Delta_i} \langle c_i \, , \, \psi_{j,n} \rangle \, \psi_{j,n} \; .$$

Original flow c'_i $\Rightarrow$ Approximated flow $\tilde{c}'_i$





The error $\|c_i - c'_i\|$ must be adjusted to the detection scale 2^l.

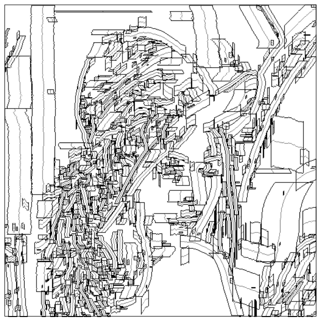
Bandelet Representation

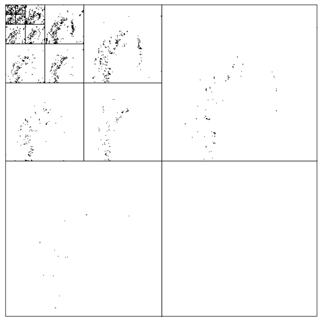
$f \quad = \quad g \quad + \quad r$

=

+

Transform Domain





bandelet

wavelet

M-Term Approximation

• M_b^i: number of bandelet coefficients in Ω_i largerer than Δ.
• M_g^i: number of wavelet coeffients of c_i larger than Δ_i.
• M_w: number of 2D wavelet coefficients of the residue larger than Δ.

For a fixed error $\|f - f_M\|$ we want to minimize

$$M = \sum_i \left(M_g^i + M_b^i \right) + M_w \ .$$

• For each tube Ω_i we find the threshold Δ_i which minimizes $M_g^i + M_b^i$.

Approximation of Piecewise Regular Images

Theorem: Suppose that f is $\mathbf{C}^\alpha$ in $[0,1]^2 - \{\bar{c}_i\}_{1\leq i\leq I}$ where the $\bar{c}_i(x)$ are $\mathbf{C}^s$ curves at a distance larger than $d > 0$. If f has discontinuities across the $\bar{c}_i$ of amplitude between $a_{\min} > 0$ and $a_{\max}$ then

$$\|f - f_M\|^2 \leq O(M^{-s}) \ .$$

This result improves a homogeneous wavelet approximation scheme for which

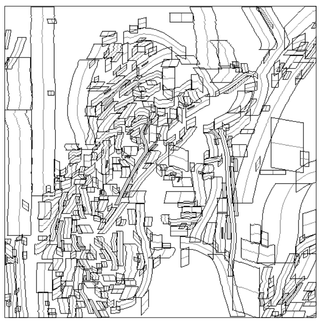
$$\|f - f_M\|^2 = O(M^{-1}) \ .$$

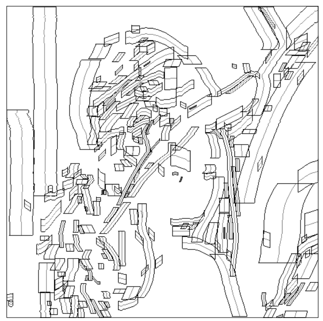
Bandelet versus Wavelet Allocation

• M_b^i: number of bandelet coefficients above Δ in Ω_i.
• M_g^i: number of wavelet coefficients of c_i above Δ_i in Ω_i.
• $M_{w,r}^i$: number of homogeneous wavelet coefficients of the residue r above Δ in Ω_i.
• $M_{w,f}^i$: number of homogeneous wavelet coefficients of the image f above a threshold Δ in Ω_i.

A bandelet representation improves the homogeneous wavelet representation of f in Ω_i if $M_g^i + M_b^i + M_{w,r}^i \leq M_{w,f}^i$.
If not, it is replaced by a homogeneous wavelet representation.

Bandelet segmentation $\Rightarrow$ Optimized Representation





PSNR = 30.42 | PSNR = 28.39

7338 Coefficients

Bandelet

Wavelet

10050 Coefficients

Wavelet

Original

PSNR = 31.06 PSNR = 30.58

4425 Coefficients

Bandelet

Wavelet

4983 Coefficients

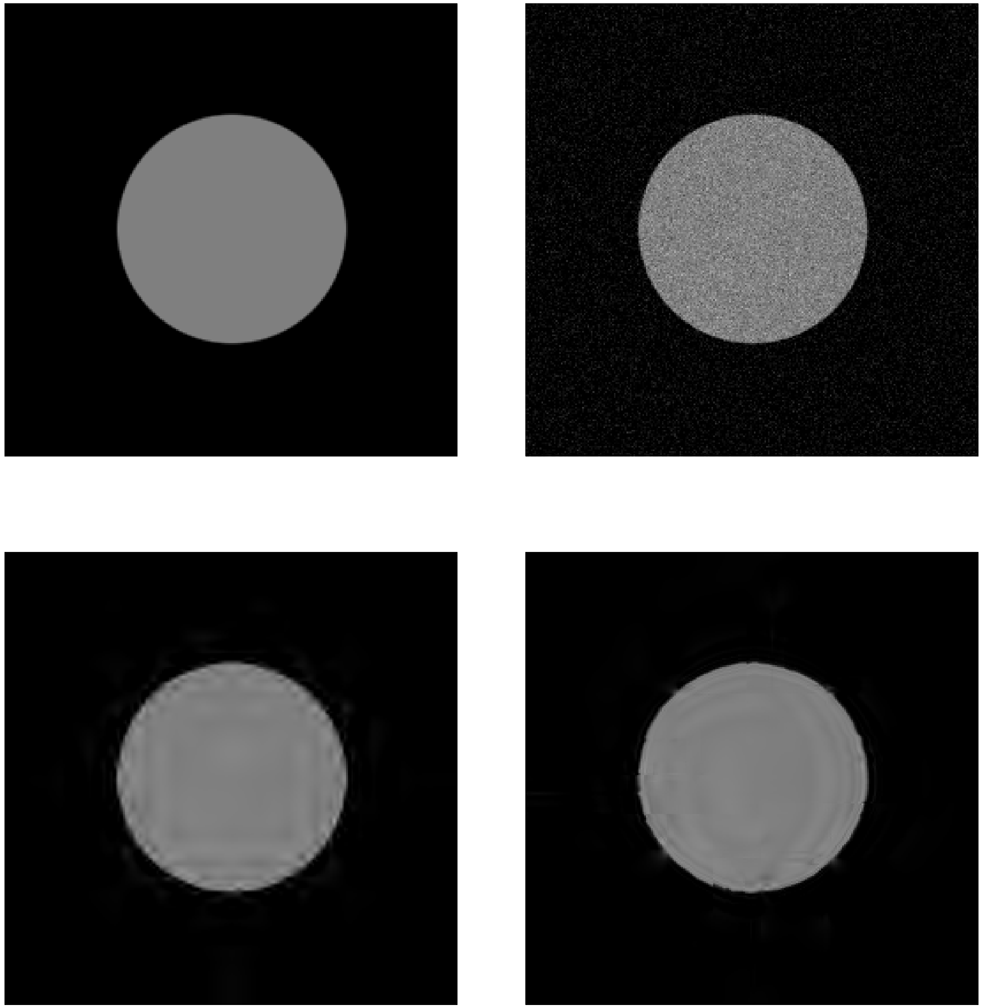
Wavelet

Original

Bandelet

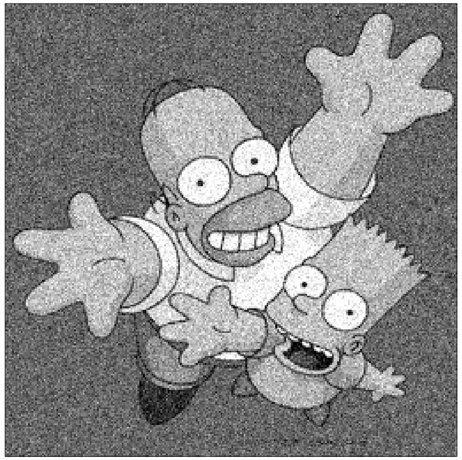
Original

Wavelet


Original
Noisy Image
PSNR = 18.6
PSNR = 29.0
PSNR = 31.6
Wavelet
Bandelet

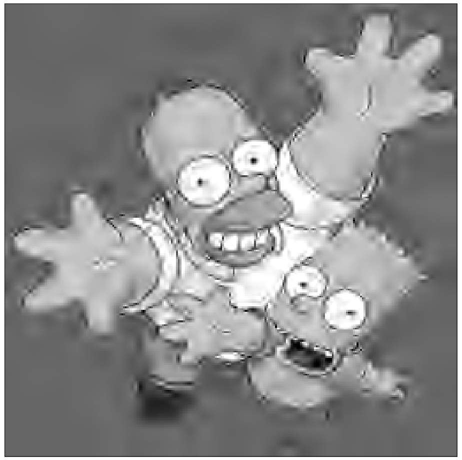
Original

Noisy Image



PSNR = 20.8

PSNR = 22.6



Wavelet

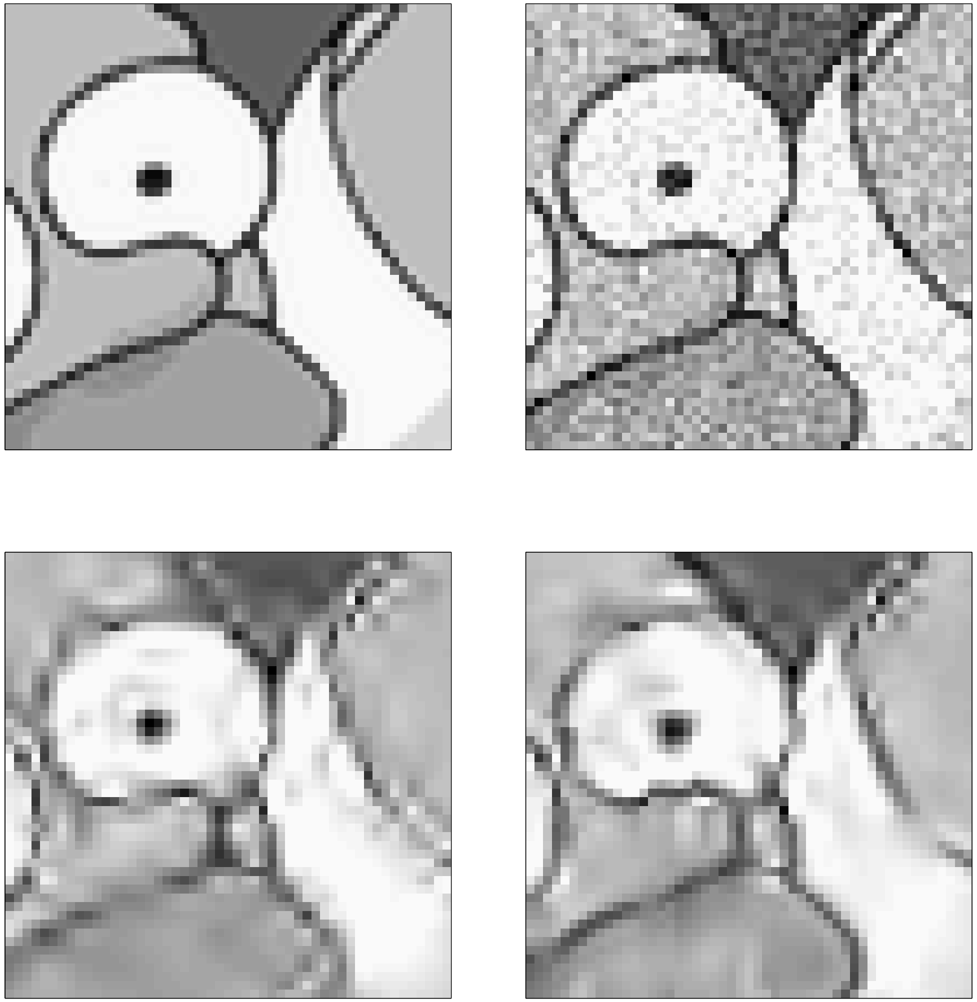
PSNR = 23.2

Bandelet

Original

Noisy Image

PSNR = 20.8



PSNR = 22.6

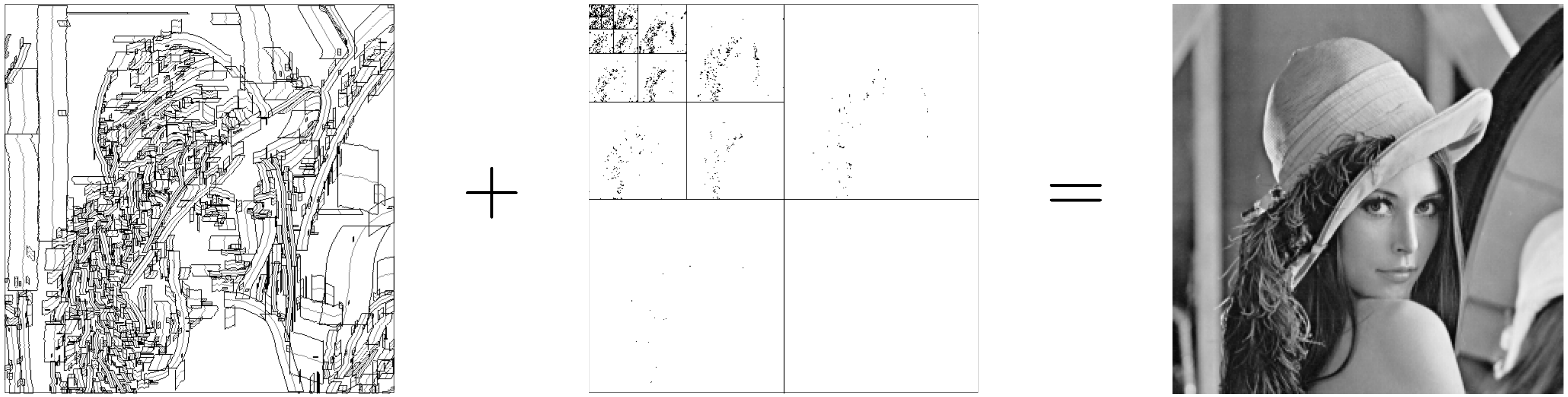
PSNR = 23.2

Wavelet

Bandelet

Compression

Replace M by a bit budget R.



- Quantize wavelet coefficients of the integral curves c_i and entropy code.
- Quantize bandelet coefficients in each tube Ω_i and entropy code.
- Quantize the 2D wavelet coefficients of the residue.
- Optimize the bit allocation by adjusting the quantizations.

Conclusion

- Bandelets provide a mathematical and algorithmic foundation to build sparse geometrical image representations in adapted bases.
- Applications to most image processing:
  - Still image coding: second generation codes.
  - Denoising and restoration by thresholding.
  - Video coding with regions.
  - Adapted to pattern recognition.