Sparse Geometrical Image Representations

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Geometrical Image Representation

Goal: Construct a *Second Generation Image Code*.

- Improve current compression rates.
- Represent the image with “features” useful for pattern analysis: Internet search...
- Mathematical and algorithmic framework to build such codes.

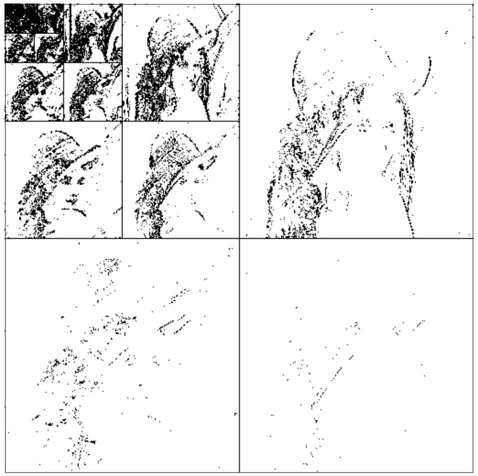
Central difficulty: extract and represent the image *geometry*.

Successes and Failures of Wavelet Bases

Representation: images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f

Significant coefficients



f_M

Limitations:

- Does not take advantage of geometric regularity.
- Not translation invariant and thus can not be used for pattern analysis.

Overview

- Foveal Wavelets for singularity.
- Bandelets for edge representation.
- Image approximations with bandelets.

Foveal Approximation

- Retina: resolution decreases from the center (fovea) to the periphery.
- Active vision: the retina moves with saccade to important locations.
- One-dimensional retina:

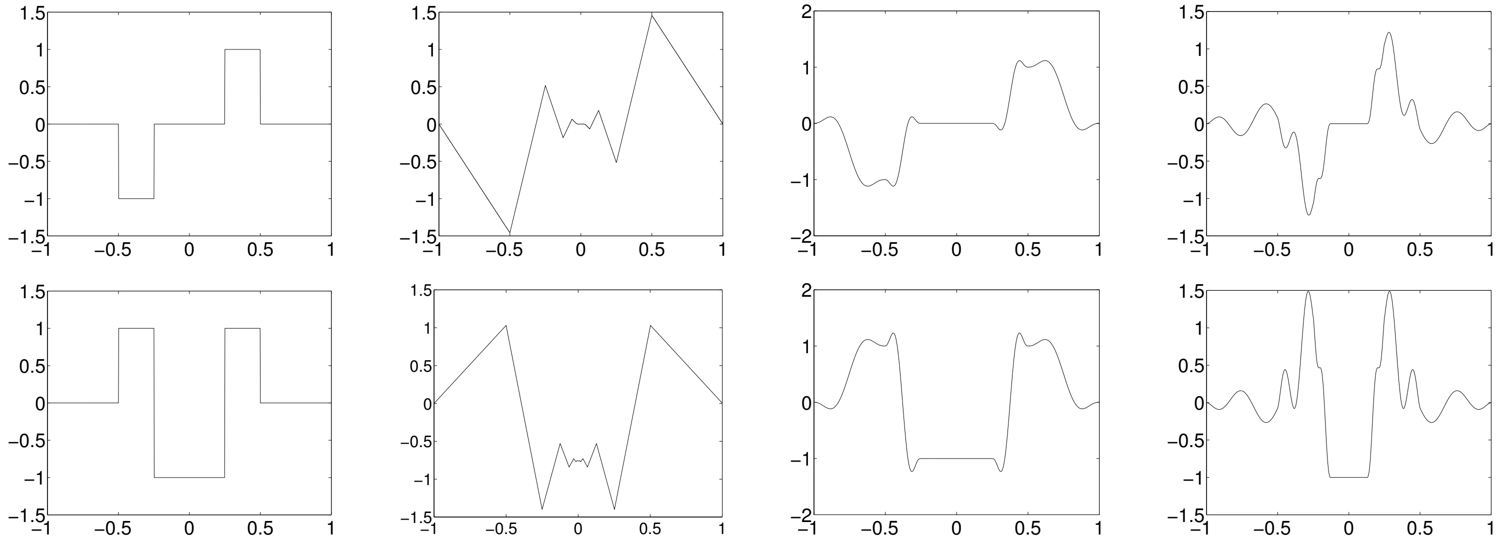
Foveal Wavelets

Theorem: There exists Ψ^1 and Ψ^2 such that

$$\left\{ \Psi^1_{j,u}(x) = \frac{1}{\sqrt{2^j}} \Psi^1\left(\frac{x-u}{2^j}\right) \ , \ \Psi^2_{j,u}(x) = \frac{1}{\sqrt{2^j}} \Psi^2\left(\frac{x-u}{2^j}\right) \right\}_{j\in\mathbb{Z}}$$

is an orthonormal basis of $\mathbf{V}_u$.

$$P_{\mathbf{V}_u} f = \sum_{k,j} \langle f, \Psi^k_{j,u} \rangle \ \Psi^k_{j,u}$$

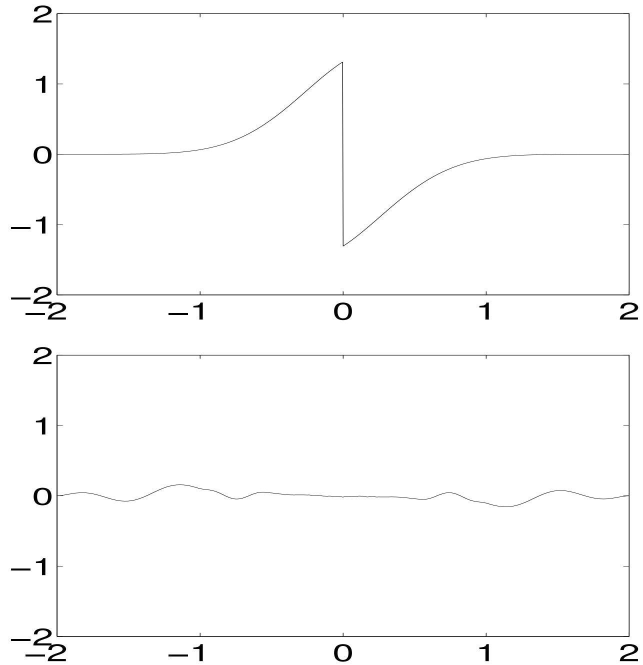


Singularity Cancellation

Theorem: Suppose that $\mathbf{V}_u$ includes polynomials of degree p.
If f is $\mathbf{C}^\alpha$ on $[a, u[$ and $]u, b]$ then

$$r = f - P_{\mathbf{V}_u} f$$

is $\mathbf{C}^\alpha$ on $[a, b]$ and $|r(x)| = O(|x - u|^\alpha)$.



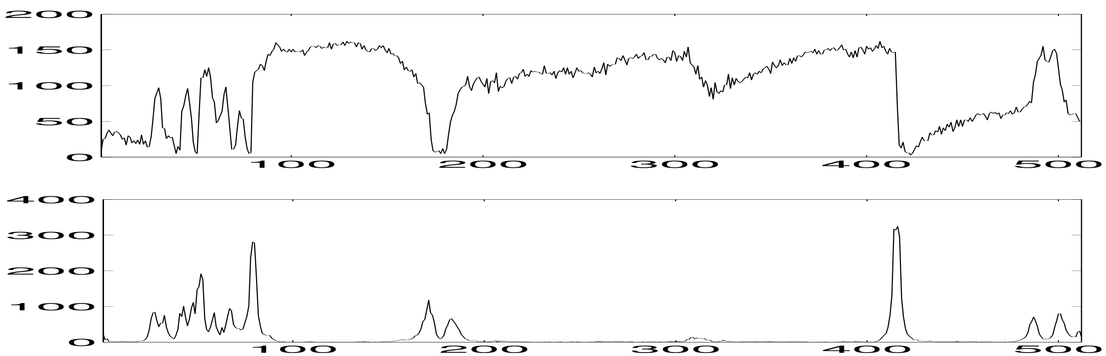
Singularity Detection

Theorem: A non-oscillating f is Lipschitz α at u for $0 \leq \alpha \leq 1$ if and only if

$$|\langle f\,,\,\Psi^1_{j,u}\rangle| = O(2^{j(\alpha+1/2)}) \text{ and } |\langle f\,,\,\Psi^2_{j,u}\rangle| = O(2^{j(\alpha+1/2)})\ .$$

Singularities are detected from the local maxima of the foveal energy

$$e(u) = \sum_j 2^{-3j}\left(|\langle f, \Psi^1_{j,u}\rangle|^2 + |\langle f, \Psi^2_{j,u}\rangle|^2\right)\ .$$

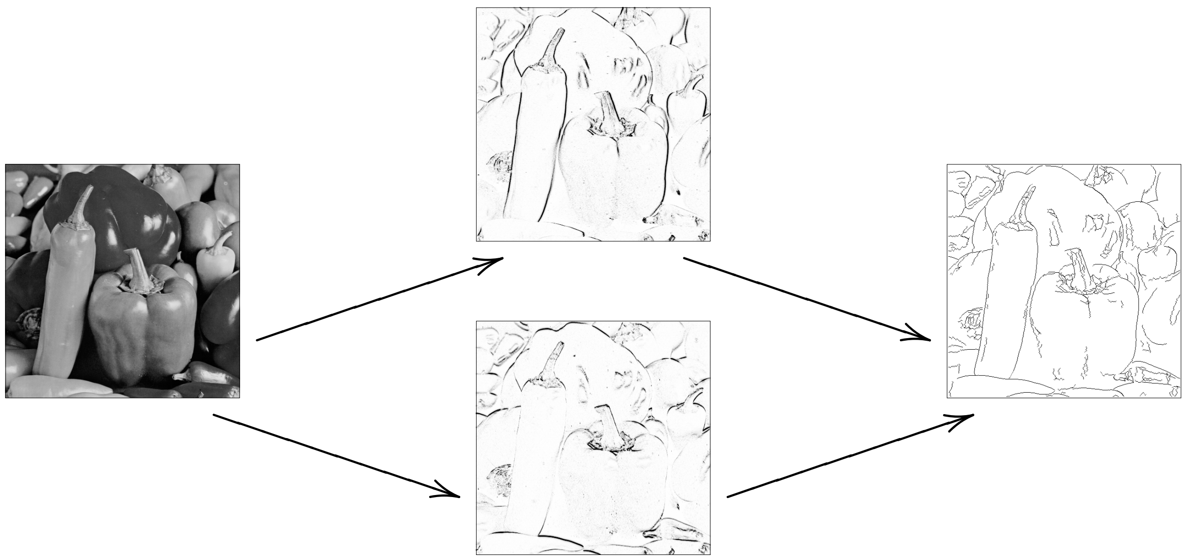


Edge Detection in Images

- Edges: one-dimensional singularities along curves.
- Canny-like detection from the 2D foveal energy

$$e(u,v) = \begin{pmatrix} e_x(u,v) \\ e_y(u,v) \end{pmatrix} \text{ where}$$

$$e_x(u,v) = \sum_j 2^{-3j} \left(|\langle f(x,v), \Psi^1_{j,u}(x) \rangle|^2 + |\langle f(x,v), \Psi^2_{j,u}(x) \rangle|^2 \right) .$$



Bandelet Bases

$$a_j^k(y) = \langle f(x,y)\,,\ \Psi^k_{j,c(y)} \rangle = \int f(x,y)\,\Psi^k_{j,c(y)}(x)\,dx\ .$$

Since $a_j^k(y)$ is smooth, it is decomposed in a classical wavelet basis

$$\left\{ \psi_{l,n}(t) = 2^{-j/2}\,\psi(2^{-j}t - n) \right\}_{l,n}$$

which yields bandelet coefficients:

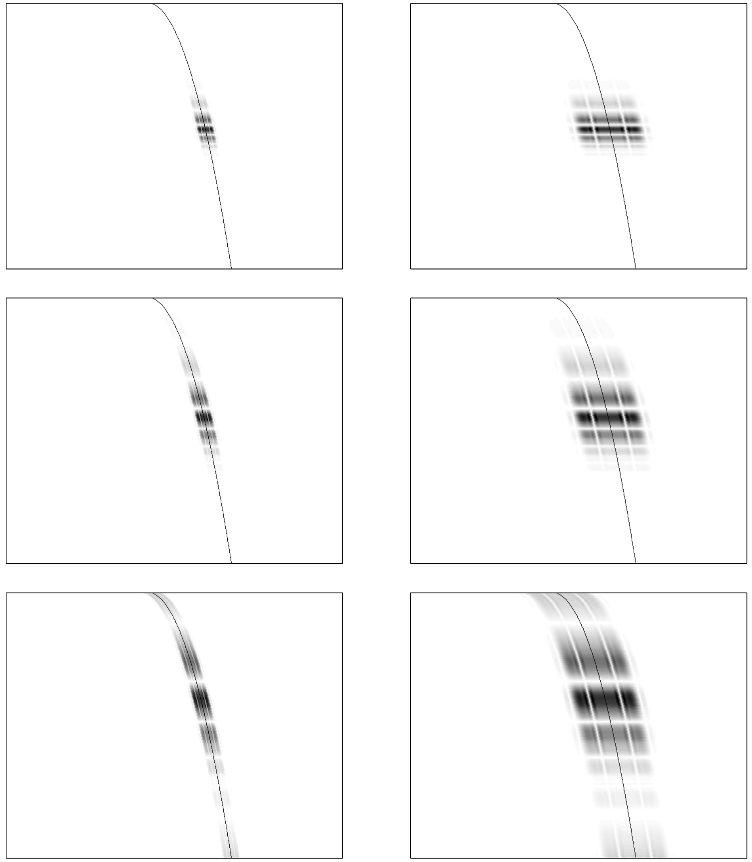
$$b^k_{j,l,n} = \langle a_j^k(y)\,,\ \psi_{l,n}(y) \rangle = \int\int f(x,y)\,\Psi^k_{j,c(y)}(x)\,\psi_{l,n}(y)\,dx\,dy\ .$$

Orthogonal bandelet basis of $\mathbf{V}_c$:

$$\mathcal{B}_c = \left\{ \Psi^1_{j,c(y)}(x)\ \psi_{l,n}(y)\ ,\ \Psi^2_{j,c(y)}(x)\ \psi_{l,n}(y) \right\}_{j,l,n}\ .$$

Bandelets

$$\Psi^{k}_{j,c(y)}(x)\ \psi_{l,n}(y)$$
$$= B_{\gamma}(x,y), \quad \gamma = (c,k,j,l,n)$$



Removal of Edges

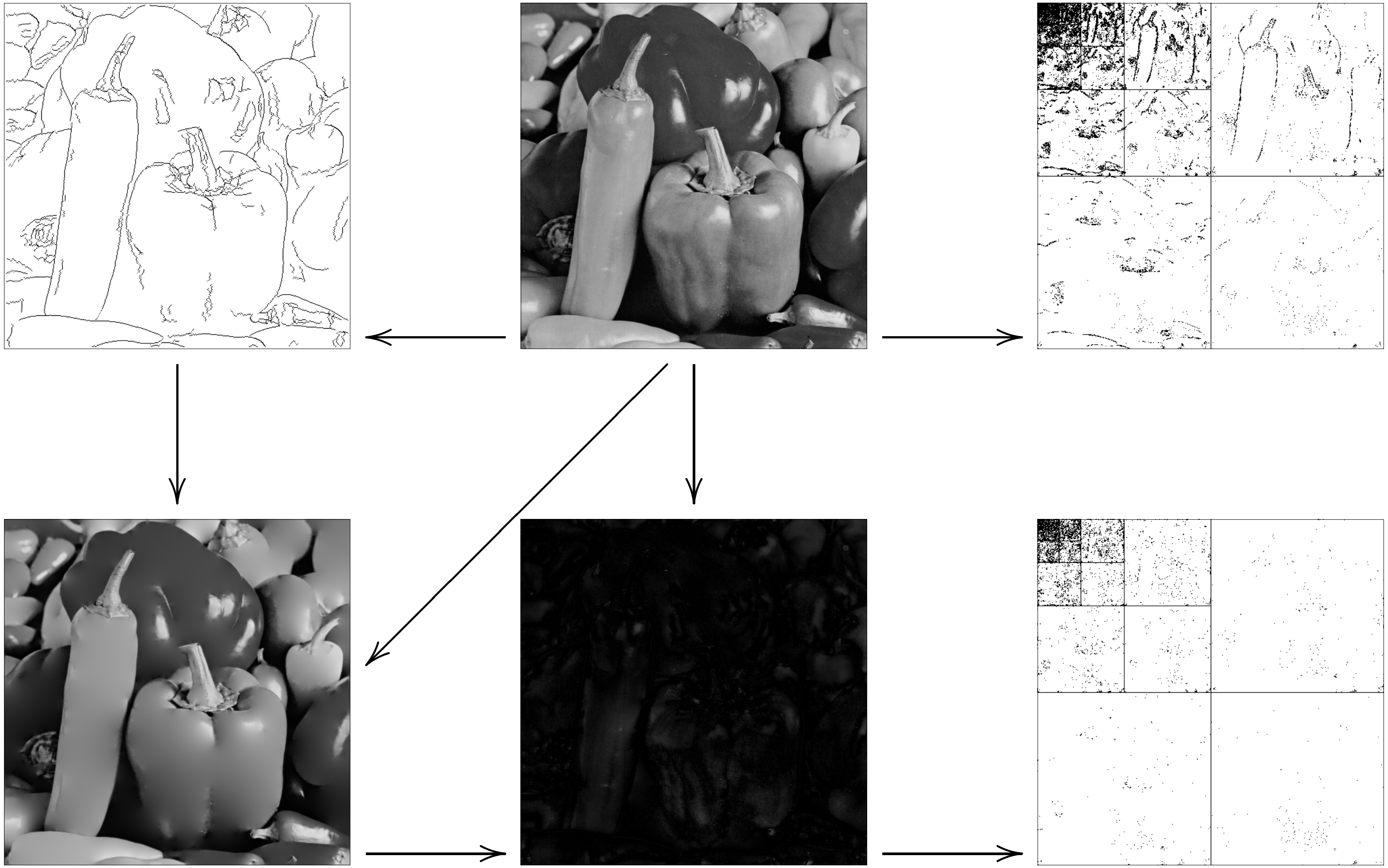
Reconstruct g from the bandelet coefficients as the solution of

$$\forall\ \gamma, \quad \langle g, B_\gamma \rangle = \langle f, B_\gamma \rangle$$

$$\text{which minimize} \iint_{I - \{c_i\}_i} |\vec{\nabla} g(x,y)|^2 \, dxdy$$

The residu $r = f - g$ is smooth.

Peppers Bandelet Decomposition

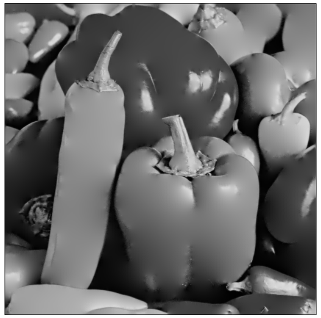


Lena Bandelet Decomposition

Bandelet Representation

$f \quad = \quad g \quad + \quad r$

=



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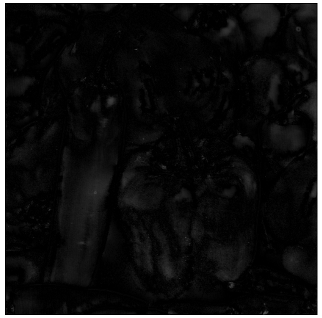
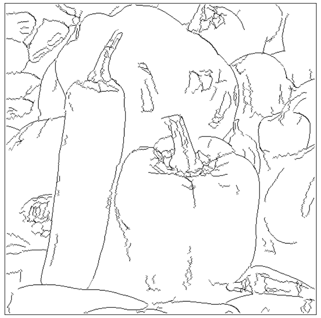
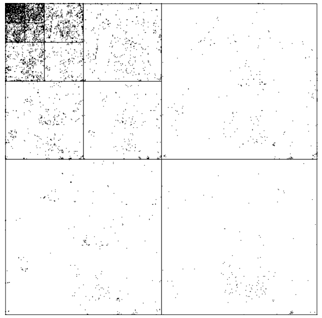


Transform Domain





bandelet

wavelet

- Approximation in tranform domains.

M-Term Approximation

- Set all coefficients below Δ to 0 :
  - Bandelet coefficients : singular profile.
  - Wavelet coefficients of r : smooth residual.

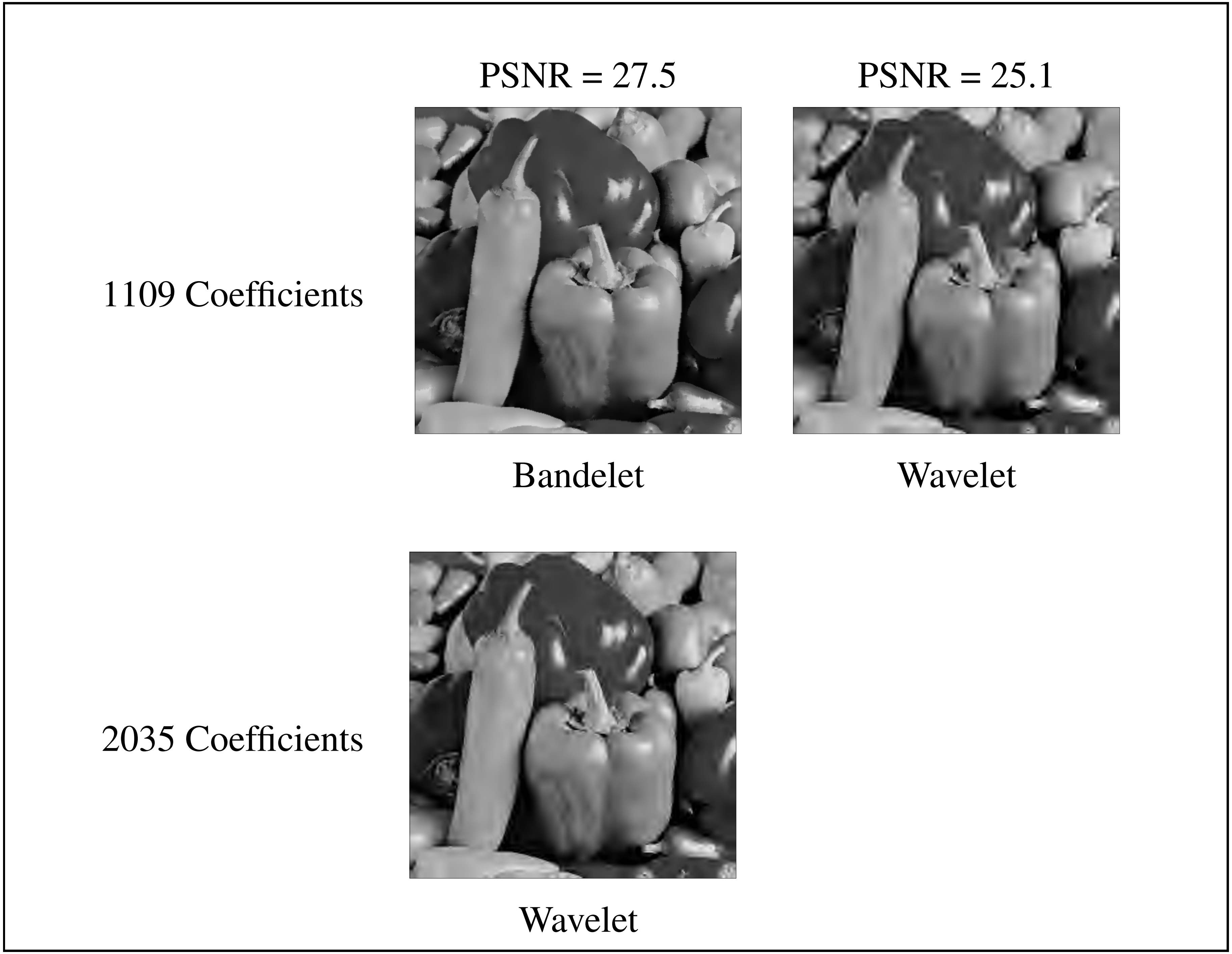
  M coefficients kept,
  Reconstruction of f_M.

- **Theorem:** If f is $\mathbf{C}^\alpha$ on $[0,1]^2 - \{c_i\}_i$ then

$$\|f - f_M\|^2 = O(M^{-\alpha}) \ .$$

- Remark : If f is approximated in a 2D wavelet basis then

$$\|f - f_M\|^2 = O(M^{-1}) \ .$$


PSNR = 27.5
PSNR = 25.1
1109 Coefficients
Bandelet
Wavelet
2035 Coefficients
Wavelet

PSNR = 27.1 PSNR = 25.6

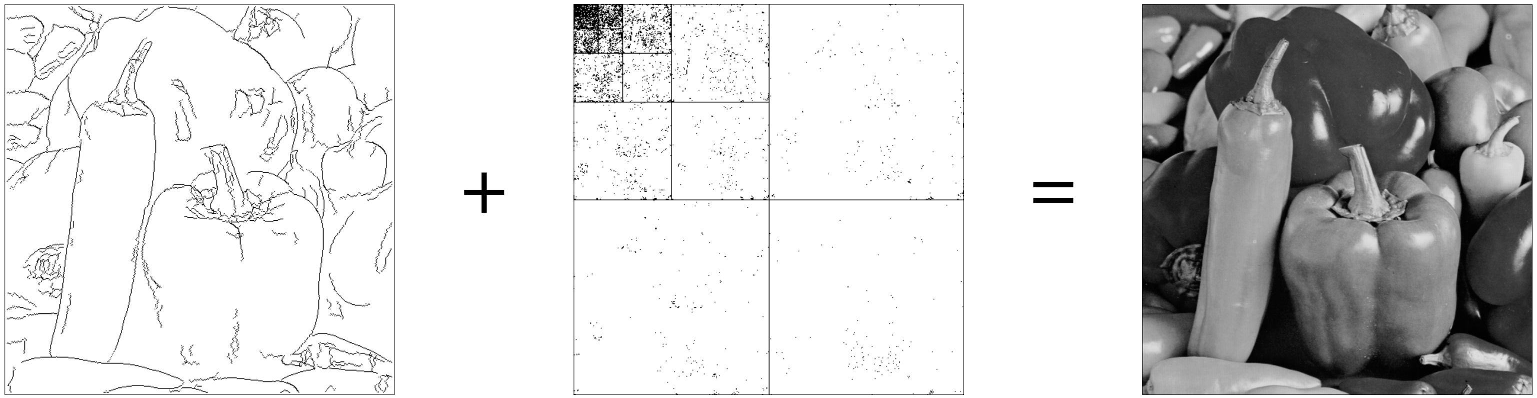
1067 Coefficients

Bandelet Wavelet

1701 Coefficients

Wavelet

Compression



- Main difficulty : optimizing the choice and the coding of the geometry .
- Pattern analysis / Search applications.

Conclusion

- Bandelets provide a mathematical and algorithmic foundation to build sparse geometrical image representations.
- A second generation image code in a transform code framework with adaptatives basis.