# Geometrical Image Representation with Bandelets

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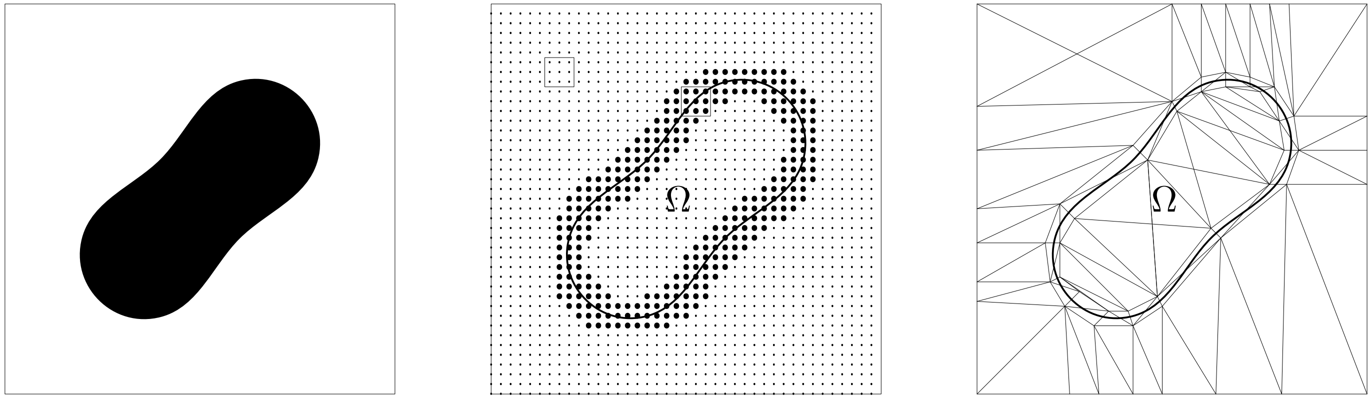
# Sparse Representations
# Using Geometric Regularity

- A representation which uses the geometric regularity along contours.
- Adapted for compression, estimation, pattern recognition, inverse problem.
- Let $f = \mathbb{1}_\Omega$, where the boundary $\partial\Omega$ is regular: $\mathbf{C^s}$ with $s > 2$.




  - Wavelet Approximation : $\|f - f_M\| \sim \|f\|_{TV}\, M^{-1/2}$.
  - With $M$ geometric elements : $\|f - f_M\| \sim C\, M^{-1}$.
  - With higer order elements : $\|f - f_M\| \sim C\, M^{-s/2}$.

# Redundant Dictionary Approximations

- Dictionary of finite elements with arbitrary geometry:

$$\mathcal{D} = \{g_\gamma\}_{\gamma\in\Gamma} \ .$$

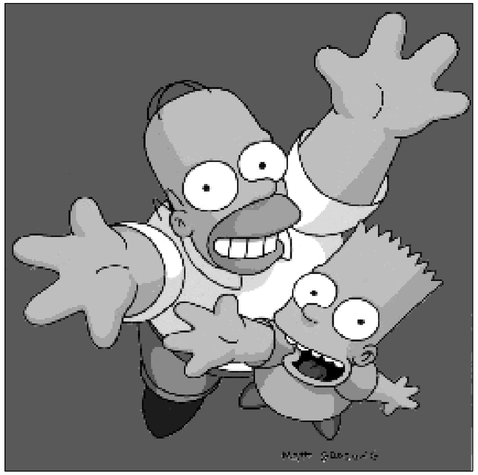
- Non-linear approximation: find $\{g_{\gamma_m}\}_{1\leq m\leq M}$ in $\mathcal{D}$ which minimize

$$\left\| f - \sum_{m=1}^{M} \alpha_m \, g_{\gamma_m} \right\| \ .$$

**Difficulties**

- Typically NP hard problem.
- Unstable solutions.
- Greedy algorithms: far from optimal.

# More Structured Approach



- Edge : One dimensional singularity that has regular evolution within image plane.
- An edge detection is needed : difficult.

# Overview

- 1D Foveal Wavelets for singularity detection.
- 2D Bandelet basis with geometry built in.
- Approximation results.

# Foveal Wavelets For Singularities

- Wavelets that zoom on singularities.
- An even wavelet $\Psi^1$ and an odd wavelet $\Psi^2$

$$\Psi^k_{j,u}(x) = \frac{1}{2^{j/2}}\, \Psi^k \left( \frac{x-u}{2^j} \right)$$

**Conditions**

- Orthonormal family of foveal wavelets

$$\left\{ \Psi^1_{j,u},\ \Psi^2_{j,u} \right\}_{j \in \mathbb{Z}}$$

- Reconstruction of polynomial discontinuities :

$$V_u = span \left\{ \Psi^k_{j,u} \right\}_{j,k}$$

$$\mathbb{1}_{(-\infty,u]} P(x) \in V_u \quad \text{and} \quad \mathbb{1}_{[u,+\infty)} P(x) \in V_u$$

if $deg(P) \leq m$

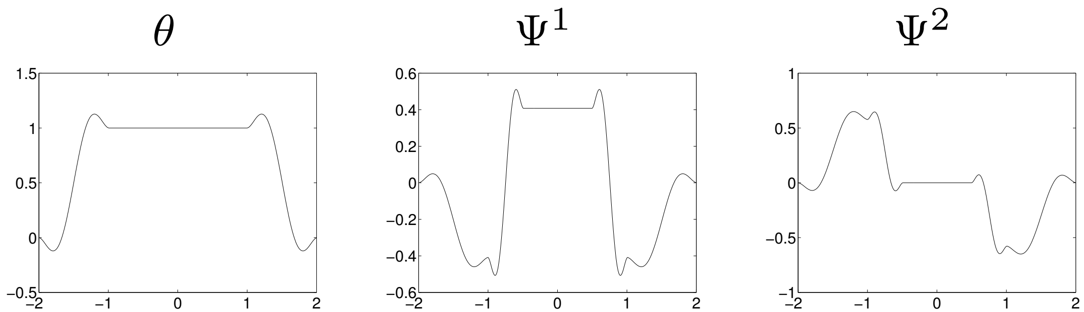
# Construction

- Foveal mother wavelets are defined by

$$\Psi^1(x) = \theta(x) - 2\theta(2x)$$

$$\Psi^2(x) = sgn(x)\,[\theta(x) - \theta(2x)]$$

With appropriate conditions on $\theta$, $\left\{\Psi^1_{j,u}, \Psi^2_{j,u}\right\}_j$ is orthonormal.



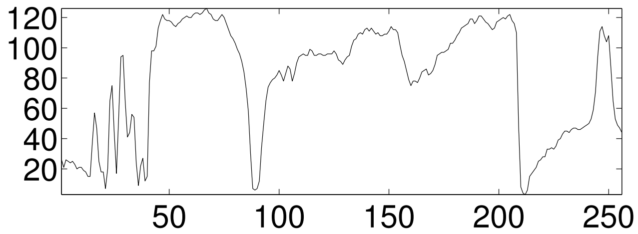
# Detection of Singularities

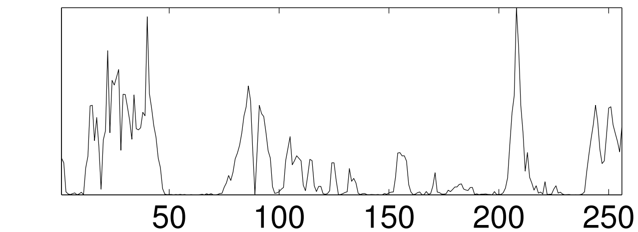
**Theorem** : A function f has a non-oscillating singularity Lipschitz of order $\alpha$ at $u$ if and only if

$$|\langle f, \Psi_{j,u}^{k} \rangle| = O(2^{j(1/2+\alpha)})$$

- To detect singularities with $\alpha < 1$, compute for each $u$

$$\epsilon(u) = \sum_{k \in \{1,2\}} \sum_{j \in \mathbb{Z}} 2^{-3j} |\langle f, \Psi_{j,u}^{k} \rangle|^2$$



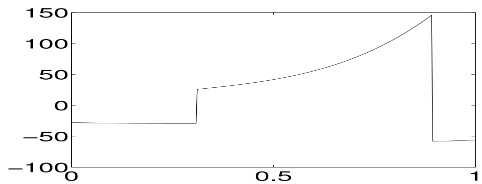
# Singularities Cancellation

**Theorem** : If is uniformly Lipschitz $\alpha < m$ for $a \leq x < u$ and $u < x \leq b$ then

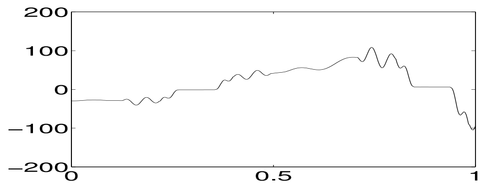
$$f - P_{V_u} f = f - \sum_{k \in \{1,2\}} \sum_{j \in \mathbb{Z}} \langle f, \Psi^k_{j,u} \rangle \; \Psi^k_{j,u}$$

is uniformly Lipschitz $\alpha$ on $[a, b]$.

Original Signal

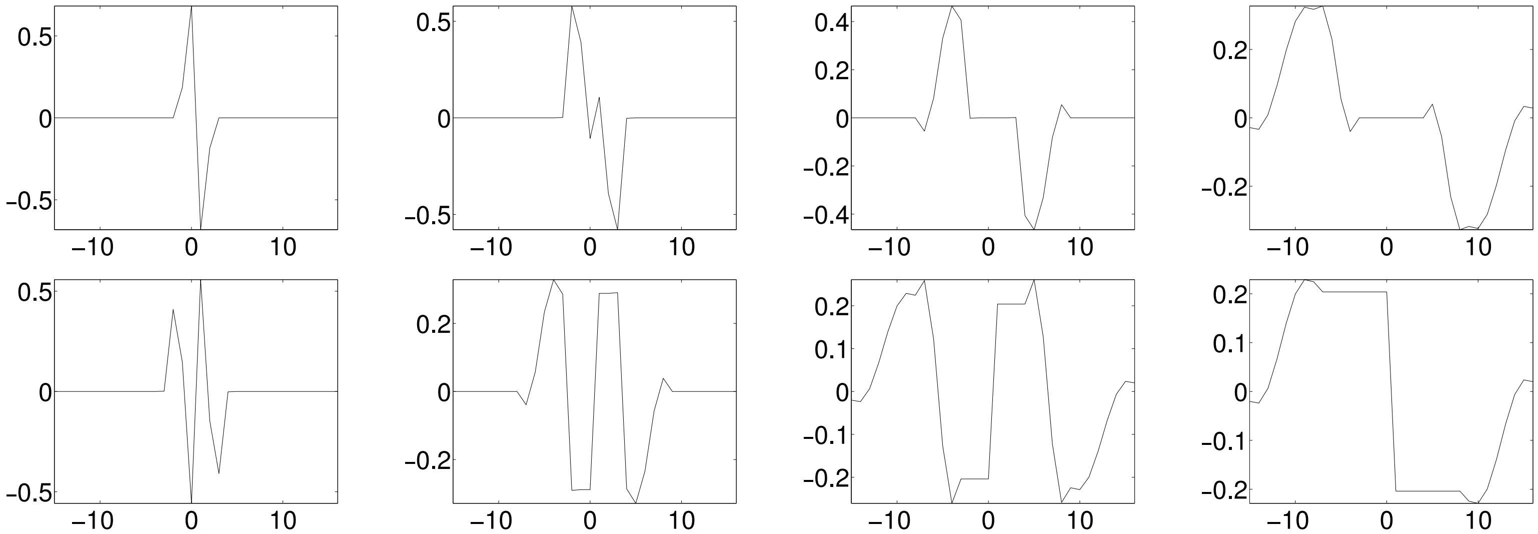



Residual $Rf = f - P_V f$ : smooth



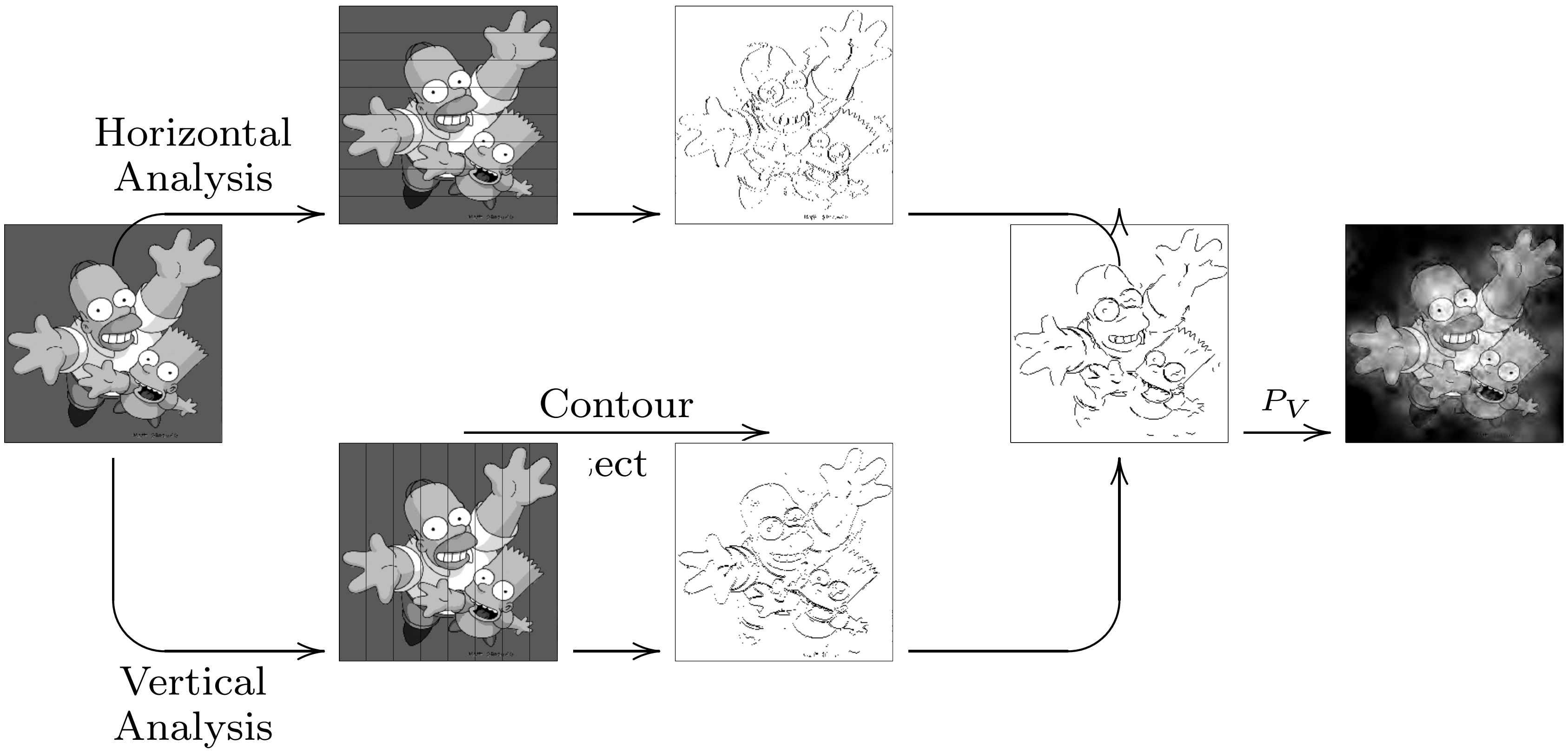
# Foveal Wavelet Packets

- Change of basis among antisymmetric foveal wavelets $\left\{\Psi^2_{j,u}\right\}$.
- Includes the projection of $sgn(x-u)$ in $V_u$.

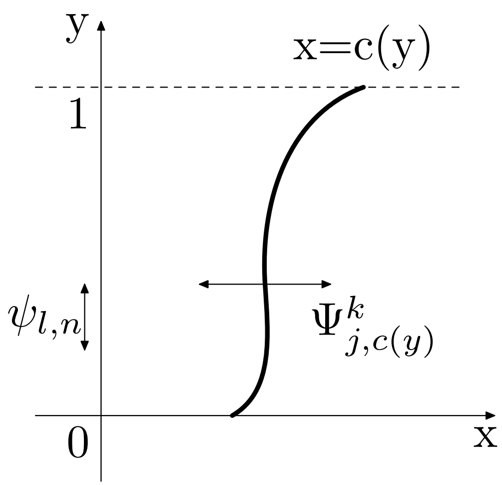


# Images and lines

- Singularities are essentially monodimensionals.
- Use only vertical and horizontal lines for detection and reconstruction.



# 2D Bandelets : geometrical basis



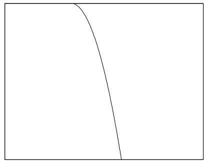

- Use a wavelet orthonormal basis

$$\left\{ \psi_{l,n} = \frac{1}{2^{l/2}} \psi \left( \frac{x - n2^l}{2^l} \right) \right\}_{l \in \mathbb{Z}^+, n \in \mathbb{Z}}$$

- Orthonormal family of bandelets

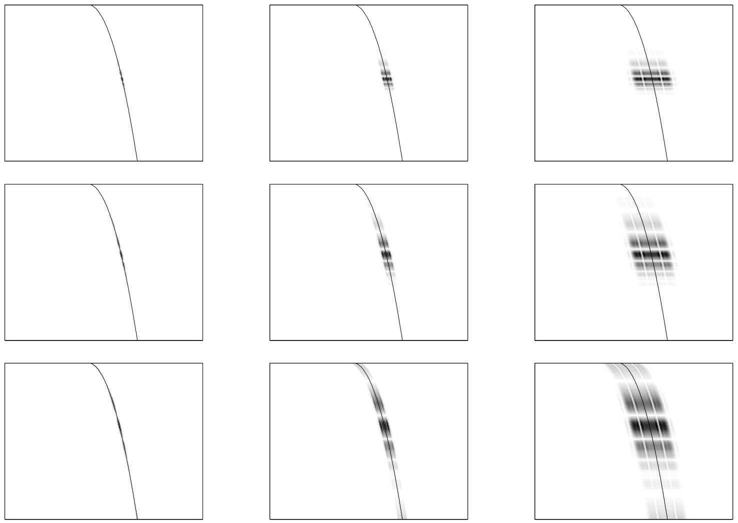
$$\mathcal{B}_c = \left\{ \Psi^1_{j,c(y)}(x)\ \psi_{l,n}(y), \quad \Psi^2_{j,c(y)}(x)\ \psi_{l,n}(y) \right\}_{j,l,n} = \{ b^c_p \}_{p \in \mathbb{Z}}$$

# Some Bandelets

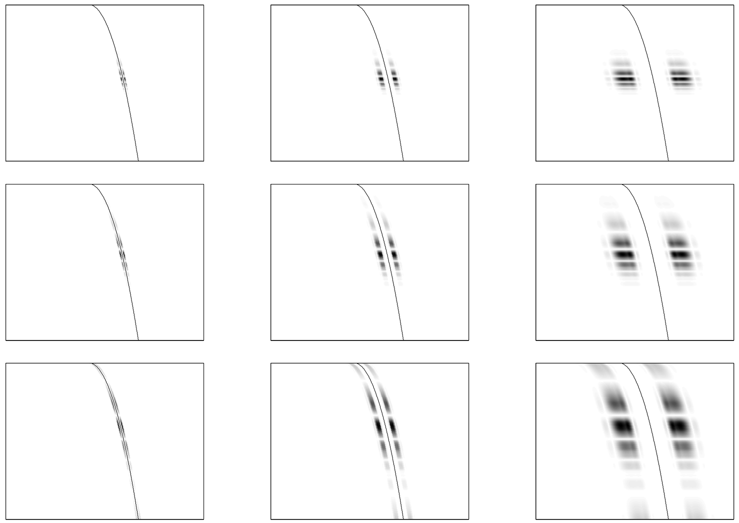


- Bandelet along the curve.

$$\Psi^1_{j,c(y)}(x)\ \psi_{l,n}(y)$$



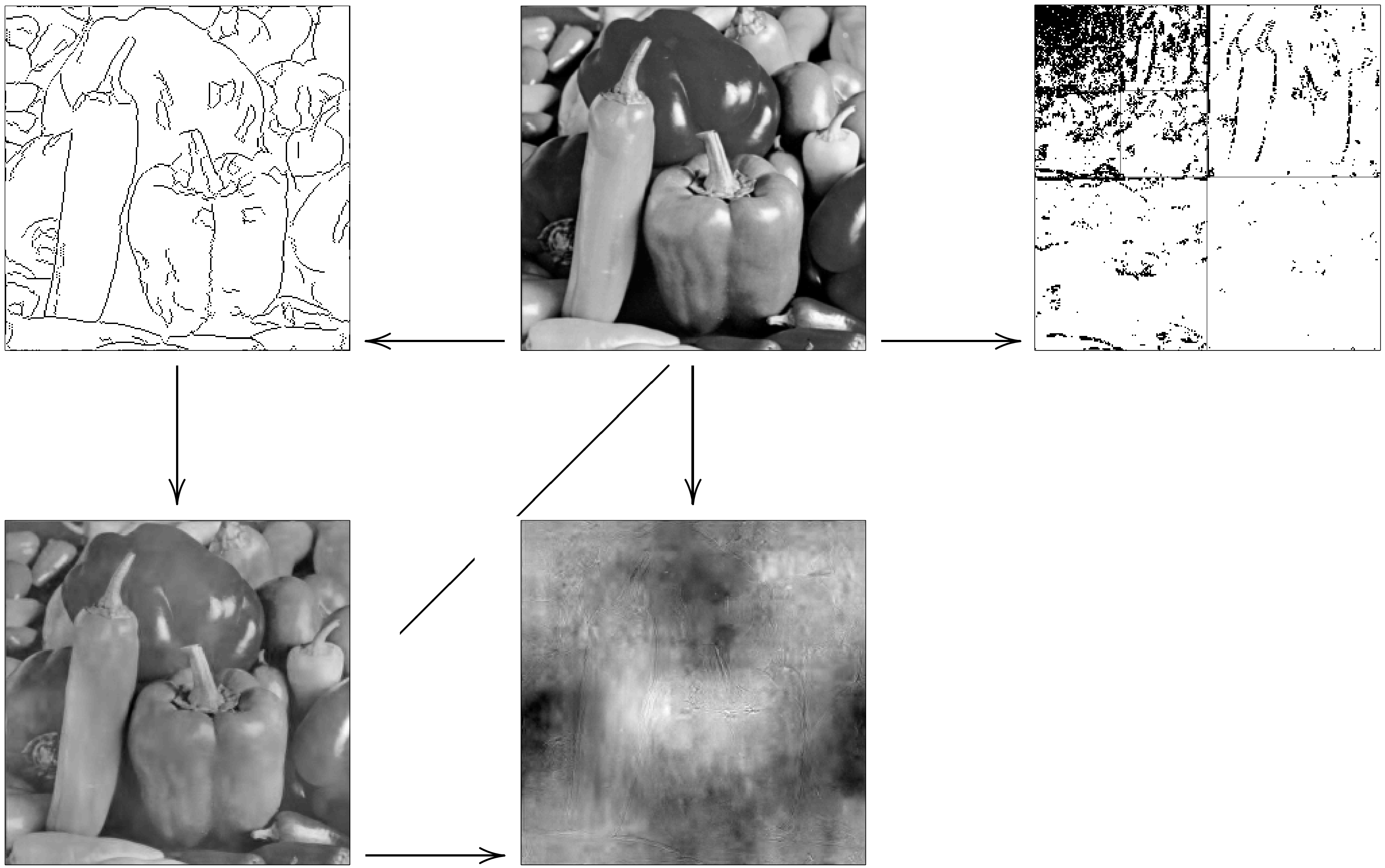
$$\Psi^2_{j,c(y)}(x)\ \psi_{l,n}(y)$$



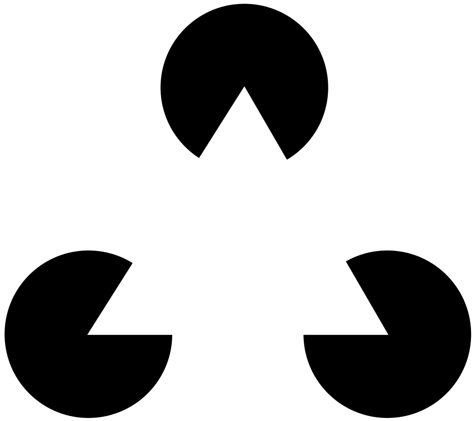
## Bandelet Frame :
## bandelet families and 2D wavelet basis

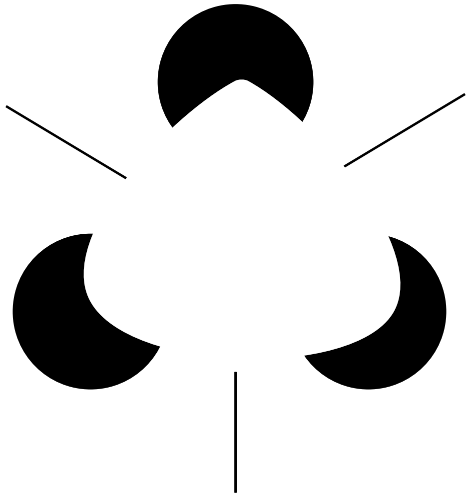
- Depends upon the choice of the $\{c_i\}$.

# Bandelet Frame Decomposition



# Where are the Edges?





- **Problem** : Find the edges $\{c_i\}_i$ which minimize $\|f - f_M\|$