IMAGE COMPRESSION WITH GEOMETRICAL WAVELETS

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ABSTRACT

We introduce a sparse image representation that takes advantage of the geometrical regularity of edges in images. A new class of one-dimensional wavelet orthonormal bases, called foveal wavelets, are introduced to detect and reconstruct singularities. Foveal wavelets are extended in two dimensions, to follow the geometry of arbitrary curves. The resulting two dimensional "bandelets" define orthonormal families that can restore close approximations of regular edges with few non-zero coefficients. A double layer image coding algorithm is described. Edges are coded with quantized bandelet coefficients, and a smooth residual image is coded in a standard two-dimensional wavelet basis.

1. GEOMETRICAL COMPRESSION

Currently, the most efficient image transform codes are obtained in orthonormal wavelet bases. For a given distortion associated to a quantizer, at high compression rates the bit budget is proportional to the number of non-zero quantized coefficients [1]. For images decomposed in wavelet orthonormal bases, these non-zero coefficients are created by singularities and contours. When the contours are along regular curves, this bit budget can be reduced by taking advantage of this regularity [2]. Many image compression with edge coding have already been proposed [3, 4, 5, 6], but they rely on ad-hoc algorithms to represent the edge information, which makes it difficult to compute and optimize the distortion rate. In this paper, we construct "bandelet" orthonormal bases that carry all the edge information and take advantage of their regularity by concentrating their energy over few coefficients. An application to image compression is studied.

2. FOVEAL WAVELET BASES

Contours are considered here as one-dimensional singularities that move in the image plane. We first construct a new family of orthonormal wavelets, all centered as the same location, which can "absorb" the singular behavior of a signal. We define two mother wavelets $\Psi^1(t)$ and $\Psi^2(t)$, which are respectively antisymmetric and symmetric with respect to $t = 0$, and such that $\int \Psi^k(t)dt = 0$ for $k = \{1, 2\}$. For any location u we denote

$$\Psi^k_{j,u}(t) = 2^{-j/2}\, \Psi^k(2^{-j}(t-u)) \quad \text{for } k = 1, 2.$$

There exists such mother wavelets, which are $\mathbf{C}^1$ and such that for any $u \in \mathbb{R}$ and $J \in \mathbb{Z}$, the family

$$\{\Psi^k_{j,u}(t)\}_{-\infty<j\leq J\,,\,k\in\{1,2\}}$$

is orthonormal [7]. These wavelets zoom on a single position u and are thus called *foveal wavelets*, by analogy with the foveal vision. To reconstruct discontinuities, we insure that left and right indicator functions, $\mathbf{1}_{[u,+\infty)}$ and $\mathbf{1}_{(-\infty,u]}$ can be written as linear combinations of foveal wavelets. This is the case for the mother wavelets shown in Figure 1. Foveal wavelets of larger support, which also reconstruct discontinuities of higher derivatives are constructed in [7], but will not be used here. Foveal wavelet families are easily discretized while retaining their orthogonality properties. The scale parameter 2^j is then limited by the resolution of the signal measurement.

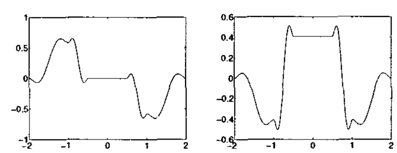

Fig. 1. Foveal mother wavelets Ψ^1 and Ψ^2

Let $\mathbf{V}_u$ be the space generated by the foveal family located at u. The orthogonal projection of f in $\mathbf{V}_u$ is

$$P_{\mathbf{V}_u} f(t) = \sum_{j=-\infty}^{J} \sum_{k=1}^{2} \langle f, \Psi^k_{j,u}\rangle \Psi^k_{j,u}(t)\,.$$

An important property of these foveal wavelets is their ability to eliminate singularities located at u. If f is differentiable in a left and right neighborhood of u, but not at u where it may be discontinuous, then one can prove that $f - P_{\mathbf{V}_u} f$ is continuous at u and has a bounded derivative

* Support in parts by an Alcatel-Espace grant and a DARPA-FastVideo grant 25-74100-F0945

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over a whole neighborhood of u [7]. The singularity of f at u is thus absorbed by its foveal coefficients at u.

Non-oscillating singularities of f are entirely characterized by the foveal wavelet coefficients at u. One can prove that f is Lipschitz α at u if and only if $|\langle f, \Psi^k_{j,u}\rangle| = O(2^{-\alpha(j+1/2)})$. Singularities are detected by computing

$$\epsilon(u) = \sum_{j=-\infty}^{J} 2^{-2j} \sum_{k=1}^{2} |\langle f, \Psi^k_{j,u}\rangle|^2 . \quad (1)$$

If f has a Lipschitz regularity $\alpha < 1$ at u, and hence is not differentiable at u then $\epsilon(u) = +\infty$, but if f is Lipschitz regularity $\alpha > 1$ at u then $\epsilon(u) < +\infty$. Singularity are thus detected from the amplitude of $\epsilon(u)$.

For signal compression, one must concentrate the signal energy over the fewest possible number of coefficients. For a signal whose singularities are mostly discontinuities, a principal component analysis over foveal wavelet coefficients shows that at any given location, most of the energy is absorbed by a single vector, which is the projection of a discontinuity is the space $\mathbf{V}_u$. To incorporate this vector among foveal wavelets, an orthogonal change of basis is performed among antisymmetric foveal wavelets $\{\Psi^1_{j,u}\}_j$. The resulting orthonormal basis $\{\tilde{\Psi}^1_{j,u}\}_j$ includes a vector that is discontinuous at u and the remaining ones have the same support but are continuous at u and have a number of oscillations that depend upon j. These vectors are very similar to wavelet packets. and are thus called *foveal wavelet packets* [8]. Figure 2 shows the graph of the discrete antisymmetric foveal wavelets at the four finest scales, and the foveal wavelet packets are constructed from these.

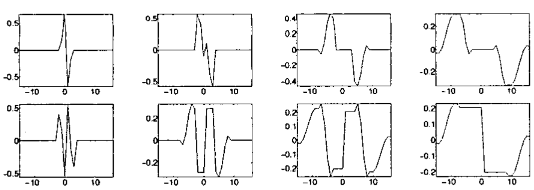

Fig. 2. The top row shows antisymmetric foveal wavelets $\Psi^1_{j,0}$ and the bottom row displays the corresponding foveal wavelet packets $\tilde{\Psi}^1_{j,0}$. The projection of discontinuity on $\mathbf{V}_0$ is at the bottom right.

3. BANDELETS FOR IMAGES

The singularities of an image $f(x_1, x_2)$ are detected with one-dimensional foveal wavelets, along each line and each column of the image. The detected singularities are chained together to form edge curves, and a two-dimensional bandelet family is constructed along each curve. These bandelets reconstruct the singular profile of the image along each edge curve.

Let us consider a horizontal scan-line defined by $f(x_1, u_2)$, where u_2 is a fixed and x_1 varies. It is decomposed over one-dimensional foveal wavelets, and for each u we compute

$$\epsilon_{u_2}(u) = \sum_{j} 2^{-2j} \sum_{k=1}^{2} |\langle f(x_1, u_2) \, , \, \Psi^k_{j,u}(x_1)\rangle|^2 .$$

A singularity corresponds to a point u_1 where $\epsilon_{u_2}(u)$ is locally maximum when u varies. This singularity is located (u_1, u_2) in the image plane. The same procedure is applied along the image columns to detect singularities. Figure 3(b) gives the value of $\epsilon_{u_2}(u)$ for an image scan-line shown in Figure 3(a).

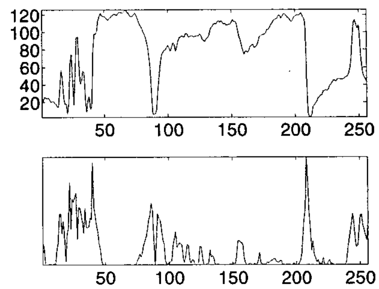

Fig. 3. (top): Horizontal scan-line of the image in Figure 5(a) at $u_2 = 170$. (bottom): Value of $\epsilon_{u_2}(u)$.

Singularities detected along lines and columns are chained together to form edge curves. These edges are either parameterized vertically with $u_1 = c(u_2)$ or horizontally with $u_2 = c(u_1)$. The singularity profile of the image along a vertical edge at (u_1, u_2) can be restored by the foveal wavelet coefficients

$$w^c_{j,k}(u_2) = \langle f(x_1, u_2) \, , \, \Psi^k_{j,c(u_2)}(x_1)\rangle. \quad (2)$$

If the edge is regular, then one can verify that for j and k fixed, the foveal coefficient $w^c_{j,k}(u_2)$ vary smoothly as a function of u_2. These coefficients are thus efficiently compressed by decomposing them along u_2 in a standard Daubechies orthogonal wavelet basis:

$$\{\psi_{l,n}(t) = 2^{-l/2}\psi(2^{-l}t - n)\}_{l,n} .$$

Observe that

$$\langle w^c_{j,k}(x_2) \, , \, \psi_{l,n}(x_2)\rangle = \langle f(x_1, x_2) \, , \, \psi_{l,n}(x_2) \, \Psi^k_{j,c(x_2)}(x_1)\rangle . \quad (3)$$

These are the decomposition of $f(x_1, x_2)$ in a family functions

$$\left\{\psi_{l,n}(x_2) \, \Psi^k_{j,c(x_2)}(x_1)\right\}_{l,n,j,k} , \quad (4)$$

which is called a *wavelet band* or *bandelets*, because their support are in a band surrounding the curve $x_1 = c(x_2)$. If the curve is parameterized by $x_2 = c(x_1)$ then the corresponding bandelet family is

$$\left\{\psi_{l,n}(x_1) \, \Psi^k_{j,c(x_1)}(x_2)\right\}_{l,n,j,k} . \quad (5)$$

From the orthogonality of foveal vavelets and Daubechies wavelets, we easily prove that a wavelet band is orthonormal [8]. Figure 4 shows few bandelets along a particular curve. Observe also that the antisymmetric foveal wavelet Ψ_j^1 can be replaced by foveal wavelet packets $\tilde{\Psi}_j^1$ without affecting the orthogonality of these families.

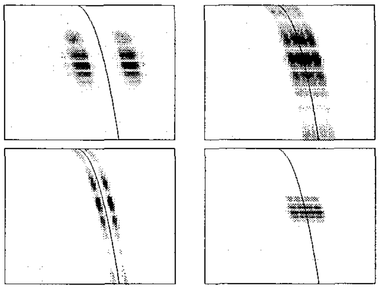

Fig. 4. Examples of bandelets in (4) along a curve $x_2 = c(x_1)$ shown in black.

The Daubechies wavelet in (3) takes advantage of the smooth evolution of the singularity profile to produce few bandelet coefficients of large amplitude at the coarser scales 2^l, whereas fine scale coefficients are negligible. Let $\mathbf{V}_c$ be the space generated by the foveal band in (4). If f is differentiable on a left and a right neighborhood of the curve $x_1 = c(x_2)$ then one can prove [8] that the residual $r = f - P_{\mathbf{V}_c} f$ is continuous along the curve and has a bounded derivative on a whole neighborhood of this curve. The bandelet coefficients thus reproduce the edge and removes the singularities.

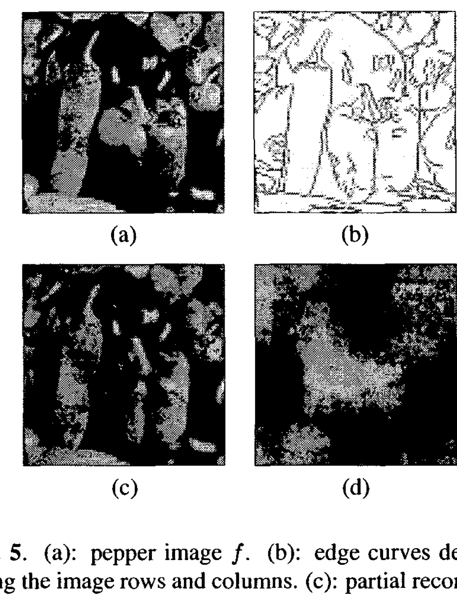

Fig. 5. (a): pepper image f. (b): edge curves detected along the image rows and columns. (c): partial reconstruction $P_{\mathbf{V}} f$ from bandelet coefficient along the edges in (b). (d): residual $r = f - P_{\mathbf{V}} f$.

Figure 5(b) shows a family of curves $\{c_i\}_i$ detected along the rows and columns of the image in Figura 5(a). Let $\mathbf{V} = \oplus_i \mathbf{V}_{c_i}$ be the space generated by the wavelet bands corresponding to all curves. Figure 5(c) displays the image $P_{\mathbf{V}} f$ reconstructed from the bandelet coefficients. It carries all the edge structures and the residual $r = f - P_{\mathbf{V}} f$ in Figure 5(d) is a smooth function.

4. IMAGE COMPRESSION

Image compression with bandelets is compared with a transform coding in a two-dimensional wavelet basis. At low bit rates, the performance of a basis in a transform code depends upon the ability of the basis to approximate the image with few non-zero coefficients [1]. The bandelet coefficients of f characterize its projection $P_{\mathbf{V}} f$, and the smooth residual r is decomposed in a two-dimensional wavelet basis. To better concentrate the image energy over few coefficients, the bandelets are constructed with foveal wavelet packets $\tilde{\Psi}_j^1$ rather than foveal wavelets Ψ_j^1. We denote f_M the signal reconstructed only from the M largest bandelet coefficients of f and wavelet coefficients of r (sorted together). For a piecewise regular image composed of regions where f is uniformly Lipschitz α, separated by boundaries curves which are also uniformly Lipschitz α, one can prove that $\|f - f_M\| = O(M^{-\alpha})$. This approximation error when decomposing directly f in a two-dimensional wavelet basis and reconstructing an approximation f_M from the M largest coefficients. For a piecewise regular image the error has a slower decay $\|f - f_M\| \sim M^{-1/2}$ because many large wavelet coefficients are needed to restore the discontinuities.

To construct an image code, the bandelet coefficients of f and the wavelet coefficients of the residual r are uniformly quantized and these coefficients are stored with an arithmetic code. The geometry of each curve c_i is recorded with a lossless wavelet lifting code [9] which takes advantage of the the regularity of these curves.

There are two potential sources of gain with respect to a transform code in an two-dimensional wavelet basis. First, we already saw that a more precise image can be reconstructed with fewer coefficients in a bandelet representation. Second, a large part of the bit budget in a wavelet code is used to code the position of the M significant coefficients (not quantized to zero). In a bandelet representation, significant coefficients are along regular curves that can be coded with fewer coefficients by taking advantage of the geometrical regularity of these curves. Numerical comparisons between JPEG-2000 and a bandelet image code will be presented at the conference.

At low bit rates, a large bit budget is affected to the coding of the geometry of edge curves as opposed to the value of bandelet coefficients. This geometrical bit budget can be reduced with a lossy code which approximates each curve $c_i(t)$ with a close curve $\tilde{c}_i(t)$. This introduce an error term in the mean-square distortion, which can be proved to be proportional to $\int |c_i(t) - \tilde{c}_i(t)|\, dt$. For piecewise regular images, a careful study of the distortion rate shows that the geometry remains the most expensive element to code, because displacements of edges introduce produce a large amplitude mean-square error. This issue is studied in [8]. However, from a perception point of view, small displacement of edges hardly affect the image quality, as long as the global shape of each edge is retained. Optimizing a bandelet code by taking advantage of visual perception is a promising direction. Examples of image compression with errors on the geometry will be shown at the conference.

5. CONCLUSION

To optimize a bandelet geometrical code, the main difficulty is to choose appropriately the curves c_i. These curve must correspond to singularities that are better coded with bandelets than with standard separable wavelets. Adapting the geometry to improve the code is a complex problem that is currently being studied.

Clearly, a bandelet code is particularly well adapted to piecewise regular images such as the peppers in Figure 5. However, a proper optimization of the double layer structure of this code, using bandelets and two-dimensional orthogonal wavelets, can guarantee a performance at least equal to a standard wavelet code, for any type of image. The allocation of bits between the geometry, the bandelet coefficients and the orthogonal wavelet coefficients of the residual remains to be better understood.

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