Reinforcement Learning

Erwan Le Pennec
Erwan.Le-Pennec@polytechnique.edu

M2DS - Reinforcement Learning – Winter 2024-2025

Outline

Outline

Decision or Decisions

Source: W. Powell

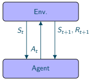
Sequential Decision Setting

Sequential Decision Setting

- In many (most?) settings, not a single decision but a sequence of decisions.
- Need to take into account the (not necessarily immediate) consequences of the sequence of decisions/actions rather than of each decisions.
- Different framework than supervised learning (no immediate feedback here) and unsupervised learning (well defined goal here).

Source: W. Powell

From Sequential Decision to Reinforcement Learning



Sequential Decision

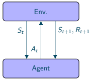
Sequential Decision

- Sequence of action A_t as a response of an environment defined by a state S_t
- Feedback through a reward R_t

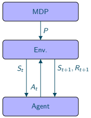
Actions?

- Is my current way of choosing actions good?
- How to make it better?

From Sequential Decision to Reinforcement Learning



Sequential Decision



MDP Modeling

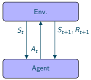
Markov Decision Process Modeling

- Specific modeling of the environment.
- Goal as as a (weighted) sum of a scalar reward.

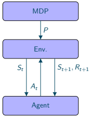
Actions?

- Is my current way of choosing actions good?
- How to make it better?

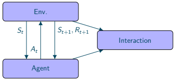
From Sequential Decision to Reinforcement Learning



Sequential Decision



MDP Modeling



Reinforcement Learning

Reinforcement Learning

- Same modeling...
- But no direct knowledge of the MDP.

Actions?

- Is my current way of choosing actions good?
- How to make it better?

Sequential Decision Settings

Sequential Decisions

- MDP / Reinforcement Learning:

$$\max_{\pi} \mathbb{E}_{\pi}\left[\sum_{t} R_t\right]$$

- Optimal Control:

$$\min_{u} \mathbb{E}\left[\sum_{t} C(x_t, u_t)\right]$$

Related settings...

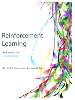
- (Stochastic) Search:

$$\max_{\theta} \mathbb{E}[F(\theta, W)]$$

- Online Regret:

$$\max \sum_{k} \mathbb{E}[F(\theta_k, W)]$$

References



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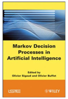

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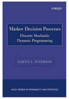

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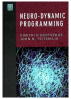

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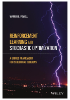

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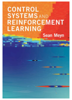

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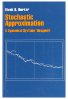

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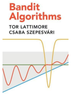

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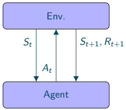
Outline

Environnement

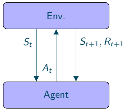


Decision Process and Environment

- At time step $t \in \mathbb{N}$:
  - State $S_t \in \mathcal{S}$: representation of the environment
  - Action $A_t \in \mathcal{A}(S_t)$: action chosen
  - Reward $R_{t+1} \in \mathcal{R}$: instantaneous reward
  - New state S_{t+1}

- Focus on the discrete setting, i.e. $\mathcal{S}$ finite, $\mathcal{A}(s)$ finite and $\mathcal{R}$ finite.
- Extension: Non finite bounded $\mathcal{R}$: easy / Non finite $\mathcal{S}$: hard / Non finite $\mathcal{A}$: harder.

Environnement



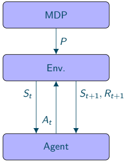
Stochastic Model

- Dynamic defined by:

$$\mathbb{P}\big(S_{t+1} = s', R_{t+1} = r \big| (S_{t'}, A_{t'}, R_{t'}), t' \leq t\big)$$
$$= \mathbb{P}\big(S_{t+1} = s', R_{t+1} = r \big| S_t = s, A_t = a, H_t\big)$$

where $H_t = (R_t, S_{t-1}, A_{t-1}, R_{t-1}, S_{t-2}, \dots)$ is the past and (S_t, A_t) the present.

Markov Decision Process and Environment



Markovian Environment

- Markovian Dynamic Assumption: S_{t+1} and R_{t+1} are independent of the past $H_t = (R_t, S_{t-1}, A_{t-1}, R_{t-1}, S_{t-2}, \dots)$ conditionally to the present (S_t, A_t).
- Dynamic entirely defined by state-reward transition probabilities

$$\begin{aligned}\mathbb{P}(S_{t+1} = s', R_{t+1} = r \middle| S_t = s, A_t = a, H_t) &= \mathbb{P}(S_{t+1} = s', R_{t+1} = r \middle| S_t = s, A_t = a)\\ &= p(s', r|s, a)\end{aligned}$$

in the discrete setting.

- Informally, this means that S_t encodes all the information related to the past.

Markov Decision Process and State-Action

- State-Reward transition probabilities for a given state-action:

$$\mathbb{P}(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a, H_t) = \mathbb{P}(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a)$$
$$= p(s', r|s, a)$$

Induced State-action laws

- State transition probabilities for a given state-action:

$$\mathbb{P}(S_{t+1} = s' | S_t = s, A_t = a, H_t) = \mathbb{P}(S_{t+1} = s' | S_t = s, A_t = a)$$
$$= p(s'|s, a) = \sum_r p(s', r|s, a)$$

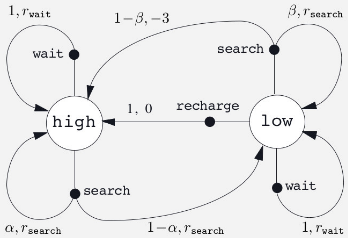
- Expected reward for a given state-action:

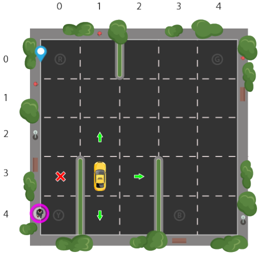
$$\mathbb{E}[R_{t+1} | S_t = s, A_t = a, H_t] = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$$
$$= r(s, a) = \sum_r r \sum_{s'} p(s', r|r, a)$$

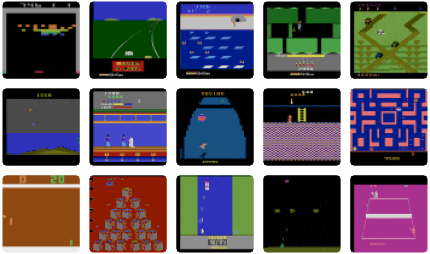
- From now on, we will always assume that the Markovian property holds for the environment.

Examples

s	a	s'	$p(s' \mid s, a)$	$r(s, a, s')$
high	search	high	α	$r_{\texttt{search}}$
high	search	low	$1-\alpha$	$r_{\texttt{search}}$
low	search	high	$1-\beta$	-3
low	search	low	β	$r_{\texttt{search}}$
high	wait	high	1	$r_{\texttt{wait}}$
high	wait	low	0	-
low	wait	high	0	-
low	wait	low	1	$r_{\texttt{wait}}$
low	recharge	high	1	0
low	recharge	low	0	-

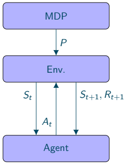






Sources:: Sutton and Barto / learndatasci.com / danijar.com

Decision Process, Agent and Policy



Agent

- Interact with the environment by choose the action given the past.

Policy Π : specification of how to choose the action

- General stochastic policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$:
$$\Pi_t(A_t = a | S_t = s, H_t) = \pi_t(A_t = a | S_t = s, H_t)$$
- General deterministic policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ (with as slight abuse of notation):
$$\Pi_t(A_t = a | S_t = s, H_t) = \mathbf{1}_{A_t = \pi_t(S_t = s, H_t)}$$

Markov Decision Process, Agent and Policy

Agent

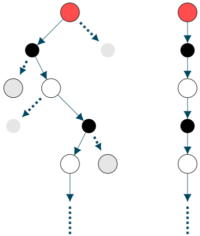
- Interact with the environment by choose the action given the past.

Policy Π: specification of how to choose the action

- History dependent stochastic policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$:
$$\Pi_t(A_t = a | S_t = s, H_t) = \pi_t(A_t = a | S_t = s, H_t)$$
- Markovian stochastic policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$:
$$\Pi_t(A_t = a | S_t = s, H_t) = \pi_t(A_t = a | S_t = s) = \pi_t(a|s)$$
- Stationary Markovian stochastic policy $\Pi = (\pi, \pi, \ldots, \pi, \ldots)$:
$$\Pi_t(A_t = a | S_t = s, H_t) = \pi(A_t = a | S_t = s) = \pi(a|s)$$

- Similar deterministic policy definition.
- Partially Observed Markov Decision Process extension: the Agent has only access to a partial observation O_t at each time step... (not the focus of the lectures)

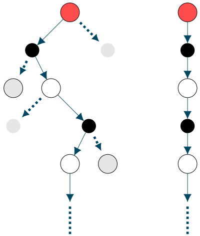
Decision Process and Trajectories



Trajectories

- Trajectory $(S_0, A_0, R_1, S_1, A_1, \ldots)$ defined by
  - an initial distribution $\mathbb{P}_0$ for S_0,
  - a policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ specifying
  $$\Pi_t(A_t = a|S_t, H_t) = \pi_t(A_t = a|S_t, H_t)$$
  - an environment specifying
  $$\mathbb{P}(S_{t+1}, R_{t+1}|S_t, A_t, H_t)$$

Decision Process and Trajectories

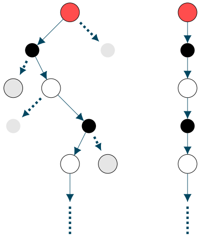


Trajectories

- Induced probability:

$$\begin{aligned}\mathbb{P}(S_0 = s_0, A_0 = a_0, R_1 = r_1, S_1 = s_1, A_1 = a_1, \dots S_t = s_t, R_t = r_t) \\ = \mathbb{P}_0(S_0 = s_0) \\ \times \pi_0(A_0 = a_0 | S_0)\, \mathbb{P}(S_1, R_1 | S_0, A_0)\, \pi_1(A_1 = a_1 | S_1 = s_1, H_1) \\ \times \cdots \times \mathbb{P}(S_t = s_t, R_t = r_t | S_{t-1} = s_{t-1}, A_{t-1} = a_{n-1}, H_{t-1})\end{aligned}$$

Markov Decision Process and Trajectories



Trajectories

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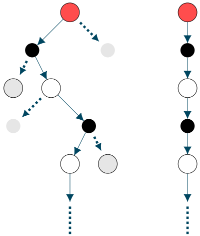
  - an initial distribution $\mathbb{P}_0$ for S_0,
  - a policy $\Pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ specifying
$$\Pi_t(A_t = a | S_t, H_t) = \pi_t(A_t = a | S_t, H_t)$$
  - a Markovian environment specifying
$$\mathbb{P}(S_{t+1}, R_{t+1} | S_t, A_t)$$

Markov Decision Process and Trajectories

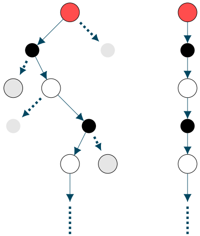


Trajectories

- Induced probability:

$$\begin{aligned}\mathbb{P}(S_0 = s_0, A_0 = a_0, R_1 = r_1, S_1 = s_1, A_1 = a_1, \dots S_t = s_t, R_t = r_t) \\ = \mathbb{P}_0(S_0 = s_0) \\ \times \pi_0(A_0 = a_0 | S_0)\, \mathbb{P}(S_1, R_1 | S_0, A_0)\, \pi_1(A_1 = a_1 | S_1 = s_1, H_1) \\ \times \cdots \times \mathbb{P}(S_t = s_t, R_t = r_t | S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1})\end{aligned}$$

Markov Decision Process and Trajectories



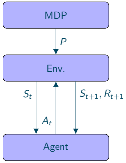
Markovian Trajectories only if the policy is Markovian

- $$\begin{aligned}&\mathbb{P}(R_{t+1}, S_{t+1}, A_{t+1}, R_{t+2}, S_{t+2}, \dots R_{t+k}, S_{t+k} | S_t, A_t, H_t)\\ &\quad = \mathbb{P}(R_{t+1}, S_{t+1}, A_{t+1}, R_{t+2}, S_{t+2}, \dots R_{t+k}, S_{t+k} | S_t, A_t)\\ &\quad = \mathbb{P}(S_{t+1}, R_{t+1} | S_t, A_t)\, \pi_{t+1}(A_{t+1} | S_{t+1})\\ &\qquad \times \cdots \times \mathbb{P}(S_{t+k}, R_{t+k} | S_{t+k-1}, A_{t+k-1})\end{aligned}$$

- Stationary if the policy is stationary.

Outline

Rewards and Total Return



Rewards and Total Returns

- MDP: Rewards R_t encode all the feedbacks!
- Quality of a policy Π measured from the remaining total return:

$$G_t = \sum_{t'=t+1}^{\infty} R_{t'}$$

- Expected total return following Π starting from s:

$$\mathbb{E}_{\Pi}[G_t | S_t = s] = \sum_{t'=t+1}^{\infty} \mathbb{E}_{\Pi}[R_{t'} | S_t = s]$$

Total Return: Issue and Fixes

Issues

- G_t is a limiting process and thus may not be defined!
- Can diverge to $\pm\infty$ or not converge at all.

Fixes?

- Finite horizon: $G_t^T = \sum_{t'=t+1}^{T} R_{t'}$
- Episodic setting: it exists a random T such that $\forall t' \geq T, R_{t'} = 0$ and $\mathbb{E}[T] < \infty$ so that $G_t = \sum_{t'=t+1}^{\infty} R_{t'}$ is well defined.
- Discounted setting: for $0 < \gamma < 1$, $G_t^\gamma = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} R_{t'}$
- Average return: $\overline{G}_t = \lim \frac{1}{T} \sum_{t'=t+1}^{t+T} R_{t'}$

Finite Horizon

$$G_t^T = \sum_{t'=t+1}^{T} R_{t'}$$

Finite Horizon Setting

- Always well defined and easy to interpret.
- Loss of Markovianity as we need to know the time step...

- Can be put in a classical Markov framework!
  - Define an absorbing state s_{abs} (a state that cannot be escaped and from which the reward is always 0).
  - Extend the state space $\mathcal{S}$ to $(\mathcal{S} \times \{0, \ldots, T\}) \cup \{s_{\text{abs}}\}$.
  - Define an state reward transition probability:

$$p\left(\tilde{s}', r | \tilde{s}, a\right) = \begin{cases} p(s', t | s, a) & \text{if } \tilde{s} = (s, t),\ t < T \text{ and } \tilde{s'} = (s', t+1) \\ 1 & \text{if } \tilde{s} = (s, t),\ t = T,\ \tilde{s'} = s_{\text{abs}} \text{ and } r = 0 \\ 1 & \text{if } \tilde{s} = s_{\text{abs}}\ ,\ \tilde{s'} = s_{\text{abs}} \text{ and } r = 0 \\ 0 & \text{otherwise} \end{cases}$$

Episodic Setting

$$G_t = \sum_{t'=t+1}^{\infty} R_{t'}$$

Episodic Setting

- Assumption: for any policy Π, the average duration before R_t remains equal to 0 is smaller than a finite horizon H: $\mathbb{E}_\Pi \left[\min_{t, R_{t'}=0, \forall t' \geq t} t \right] \leq H < +\infty$
- Strong assumption...
- Easy to interpret.

- Slightly stronger (but more convenient) def.:
  - Replace all the states from which R_t remains equal to 0 whatever the policy by a single absorbing state s_{abs},
  - Assumption: for any policy Π and any initial state, the average duration to reach this state is smaller than a finite horizon H: $\forall s, \mathbb{E}_\Pi \left[\min_{t, S_t = s_{\text{abs}}} t \middle| S_0 = s \right] \leq H < +\infty$

Discounted

$$G_t^\gamma = \sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} R_{t'}$$

Discounted

- Always defined but not that easy to interpret.
- Easiest theoretical setting!

- Equivalent to an episodic setting if one adds an absorbing state s_{abs} and changes all state-reward transition probabilities to:

$$p(s', r|s, a) = \begin{cases} \gamma p(s', r|s, a) & \text{if } s' \neq s_{\text{abs}}, s \neq s_{\text{abs}} \\ (1-\gamma) & \text{if } s' = s_{\text{abs}}, r = 0, s \neq s_{\text{abs}} \\ 1 & \text{if } s' = s_{\text{abs}}, r = 0, s = s_{\text{abs}} \\ 0 & \text{otherwise} \end{cases}$$

- Horizon $H = 1/(1-\gamma)$.

Average Return Setting

$$\overline{G}_t = \lim \frac{1}{T} \sum_{t'=t+1}^{t+T} R_{t'}$$

Average Return

- Not always defined. (Cesaro Average)
- Always equal to 0 in the episodic setting!
- Natural definition in a *stationary* setting...
- Complex theoretical analysis!

- Under a strict stationarity assumption ($R_t \sim R_{t'}$), link with discounted setting as

$$\mathbb{E}_\Pi[G_t^\gamma] = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}_\Pi[R_{t+1}] = \frac{1}{1-\gamma}\mathbb{E}_\Pi[R_t] = \frac{1}{1-\gamma}\mathbb{E}_\Pi\left[\overline{G}_t\right]$$

State Value Functions

State Value Functions

- Return expectation for a policy Π starting from s at time t
  - Finite horizon setting:
$$v^T_{t,\Pi}(s) = \mathbb{E}_\Pi\left[G^T_t | S_t = s\right] = \sum_{t'=t+1}^{T} \mathbb{E}_\Pi[R_{t'} | S_t = s]$$
  - Episodic setting:
$$v_{t,\Pi}(s) = \mathbb{E}_\Pi[G_t | S_t = s] = \sum_{t'=t+1}^{\infty} \mathbb{E}_\Pi[R_{t'} | S_t = s]$$
  - Discounted:
$$v^\gamma_{t,\Pi}(s) = \mathbb{E}_\Pi[G^\gamma_t | S_t = s] = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} \mathbb{E}_\Pi[R_{t'} | S_t = s]$$
  - Average return setting:
$$\overline{v}_{t,\Pi}(s) = \mathbb{E}_\Pi\left[\overline{G}_t | S_t = s\right] = \lim_{T \to \infty} \frac{1}{T} \sum_{t'=t+1}^{t+T} \mathbb{E}_\Pi[R_{t'} | S_t = s]$$

- Depends on t for a history dependent policy!

Markovian Policy and State Value Functions

State Value Functions

- Return expectation for a Markovian policy Π starting from s at time t.
  - Finite horizon setting (with time extended state space):
$$v^T_{t,\Pi}(s) = \mathbb{E}_\Pi\left[G^T_t | S_t = s\right] = \sum_{t'=t+1}^{T} \mathbb{E}_\Pi[R_{t'}|S_t = s]$$
  - Episodic setting:
$$v_{t,\Pi}(s) = \mathbb{E}_\Pi[G_t|S_t = s] = \sum_{t'=t+1}^{\infty} \mathbb{E}_\Pi[R_{t'}|S_t = s]$$
  - Discounted:
$$v^\gamma_{t,\Pi}(s) = \mathbb{E}_\Pi[G^\gamma_t|S_t = s] = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)}\mathbb{E}_\Pi[R_{t'}|S_t = s]$$
  - Average return setting:
$$\overline{v}_{t,\Pi}(s) = \mathbb{E}_\Pi\left[\overline{G}_t|S_t = s\right] = \lim_{T\to\infty} \frac{1}{T} \sum_{t'=t+1}^{t+T} \mathbb{E}_\Pi[R_{t'}|S_t = s]$$

- Becomes independent on t if the policy is stationary and Markovian the generic case (except in the finite horizon setting).

State-Action Value Functions

State-Action Value Functions

- Return expectation for a policy Π starting from s and an action a at time t.
  - Finite horizon setting:
$$q^T_{t,\Pi}(s,a) = \mathbb{E}_\Pi\left[G^T_t | S_t = s, A_t = a\right] = \sum_{t'=t+1}^{T} \mathbb{E}_\Pi[R_{t'} | S_t = s, A_t = a]$$
  - Episodic setting:
$$q_{t,\Pi}(s,a) = \mathbb{E}_\Pi[G_t | S_t = s, A_t = a] = \sum_{t'=t+1}^{\infty} \mathbb{E}_\Pi[R_{t'} | S_t = s, A_t = a]$$
  - Discounted:
$$q^\gamma_{t,\Pi}(s,a) = \mathbb{E}_\Pi[G^\gamma_t | S_t = s, A_t = a] = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} \mathbb{E}_\Pi[R_{t'} | S_t = s, A_t = a]$$
  - Average return setting:
$$\overline{q}_{t,\Pi}(s,a) = \mathbb{E}_\Pi\left[\overline{G}_t | S_t = s, A_t = a\right] = \lim_{T\to\infty} \frac{1}{T} \sum_{t'=t+1}^{t+T} \mathbb{E}_\Pi[R_{t'} | S_t = s, A_t = a]$$

- Different strategy for action at time t than after...
- Independent of t for a Markovian policy except for the finite horizon setting!

State Value Function vs State-Action Value Functions

$$v_{t,\Pi}(s) = \mathbb{E}_\Pi[G_t | S_t = s] \quad q_{t,\Pi}(s,a) = \mathbb{E}_\Pi[G_t | S_t = s, A_t = a]$$

State vs State-Action

- Performance measure of a policy Π:
  - starting from a state s for the state value function,
  - starting from a state s and an action a (not necessarily related to Π) for the state-action value function.
- State value function at time t from state-action value function:

$$v_{t,\Pi}(s) = \sum_a \Pi_t(a) q_{t,\Pi}(s,a)$$

Do We Really Need The History Dependent Policies?

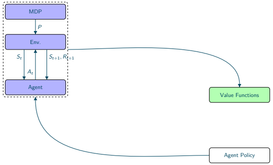
Equivalent Markovian policy in terms of value function

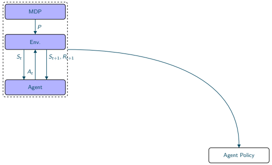
- **Thm:** For any policy Π and any initial distribution $\mathbb{P}_0(S_0)$, it exists a Markovian policy $\widetilde{\Pi}$ such that
$$\forall t, \forall s, v_{t,\Pi}(s) = v_{t,\widetilde{\Pi}}(s).$$

- Relies on the Markovian environment.
- Possible choice:
$$\widetilde{\pi}_t\left\{A_t = a_t | S_t = s_t\right\} = \mathbb{E}_{\mathbb{P},\mathbb{P}_0}[\pi_t(A_t = a_t | S_t = s_t, H_t) | S_t = s_t, S_0]$$
- **No need to consider non Markovian policy** if the goal is entirely defined in terms of value functions.

Outline

Goals





Prediction

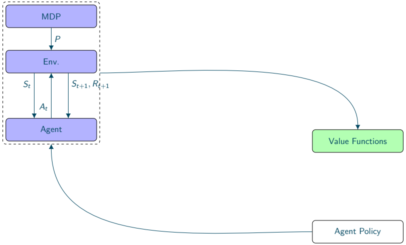
- What is the performance of a given policy?

Planning

- What is the *best* policy?

- Planning is harder than predicting.

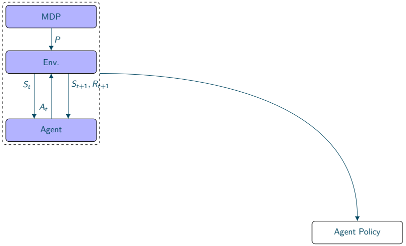
Prediction



Prediction

- What is the performance of a given policy?
- Compute/Approximate/Estimate
$$v_{t,\Pi}(s) = \mathbb{E}_\Pi[G_t | S_t = s]$$
- Well defined provided the expectation exists.

Planning

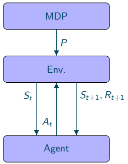


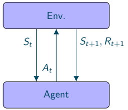
Planning

- What is the *best* policy?
- A possible definition: $\underset{\Pi}{\operatorname{argmax}} \sum_{s,t} \mu(s,t) v_{t,\Pi}(s)$
- Not necessarily well defined...
- Several choices for μ!
- More realistic goal: find a *good* policy...

Outline

What Do We Know?





Model

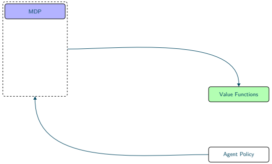
- Able to use the MDP transition probabilities.
- Markov Decision Process / Operations Research.
- Probability world.

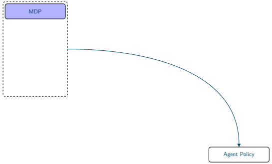
Only Observations

- No access to the MDP transition probabilities.
- Reinforcement Learning.
- Statistic world.

- Reinforcement Learning is harder than Markov Decision Process / Operations Research.

Markov Decision Process / Operations Research

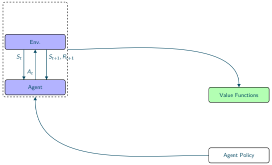


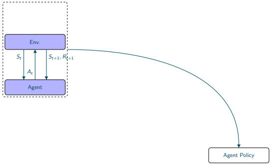


MDP / OR

- Stochastic setting in which the world is known.
- MDP model assumption.
- Probability world / Idealized setting. . .
- Lots of insight for the RL problem.

Reinforcement Learning





RL

- Stochastic setting in which the world is observed through interactions.
- Still MDP model assumption.
- More realistic setting?
- More difficult theoretical analysis.

Outline

MDP vs Discrete Control

MDP

- State s and action a
- Dynamic model:
$$s' \sim \mathbb{P}(\cdot|s, a)$$
- Reward r defined by $\mathbb{P}(r|s', s, a)$.
- Policy Π: $a_t \sim \pi_t(\cdot|S_t, H_t)$
- Goal:
$$\max \mathbb{E}_\Pi \left[\sum_t R_t \right]$$

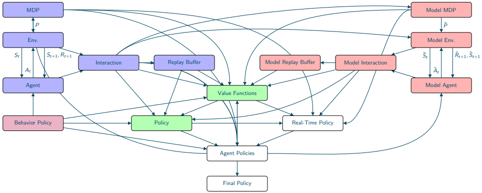
Discrete Control

- State x and control u
- Dynamic model:
$$x' = f(x, u, W)$$
with W a stochastic perturbation.
- Cost: $C(x, u, W)$.
- Control strategy U:
$u_t = u_t(x_t, H_t, W')$
- Goal:
$$\min_U \mathbb{E}_U \left[\sum_t C(x_t, u_t, W_t) \right]$$

- Almost the same setting but with a different vocabulary!

Outline

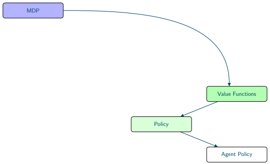
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View
- Extensions

Operations Research and MDP

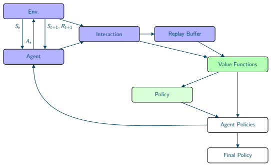


How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

Reinforcement Learning and Interactions

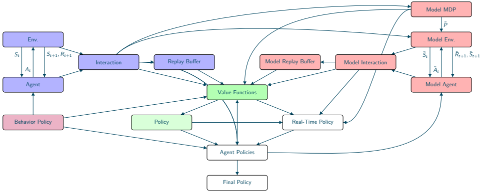


How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA. . .)
- Policy deduced by a statewise optimization of the value function over the actions.

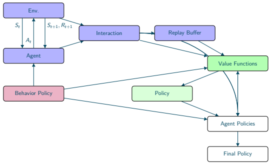
More Tabular Reinforcement Learning



Can We Do Better?

- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real-time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).

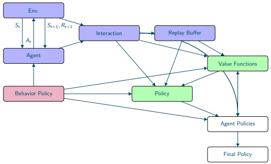


How to Deal with a Large/Infinite states/action space?

- How to approximate value functions?
- How to estimate good approximation of value functions?

- Finite action space setting.
- Stochastic algorithm (Deep Q Learning...).
- Policy deduced by a statewise optimization of the value function over the actions.

Actor/Critic: a Policy Point of View

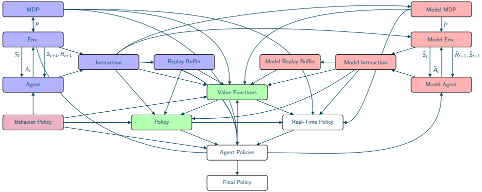


Could We Directly Parameterized the Policy?

- How to parameterize a policy?
- How to optimize this policy?
- Can we combine parametric policy and approximated value function?

- State Of The Art Algorithms (DPG,PPO, SAC...)

Extensions

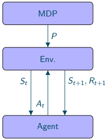


Can We Do Something Different in This Setting?

- How to deal with the total and average returns?
- How to deal with partial observations?
- How to learn a policy or an implicit reward by observing an actor?

Outline

Markov Decision Process / Operations Research



MDP / OR

- Known MDP model
- Focus on the finite horizon setting

$$G_t^T = \sum_{t'=t+1}^{T} R_{t'}$$

and the discounted setting:

$$G_t^\gamma = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} R_{t'}$$

- We will later consider the other settings.

Policy

Policy

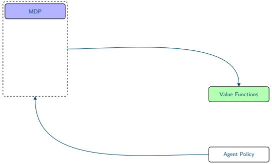
- Finite horizon : emphasis on Markovian policies

$$\Pi_t(A_t = a_t | S_t = s_t, H_t) = \pi_t(A_t = a_t | S_t = s_t) = \pi_t(a_t | s_t)$$

- Discounted return: emphasis on stationary Markovian policies

$$\Pi_t(A_t = a_t | S_t = s_t, H_t) = \pi(A_t = a_t | S_t = s_t) = \pi(a_t | s_t)$$

Prediction

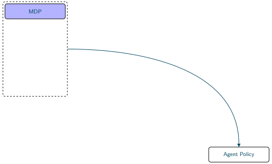


Prediction

- How to efficently evaluate the quality of a policy
$$v_{t,\Pi}(s) = \mathbb{E}_\Pi\left[\sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} R_{t'} \middle| S_t = s\right]$$
when we can ensure that the sum is finite?

- $v_{t,\Pi}$ independent of t in the discounted setting if the policy is stationary.

Planning



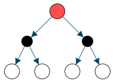
Policy

- How to find a policy π such that
$$\sum_{s,t} \mu(s,t) v_{t,\Pi}(s)$$
is as large as possible?
- Emphasis on $\mu(s,t) = 0$ if $t \neq 0$ and $\mu(s,0) = \mathbb{P}_0(S_0 = s_0)$.

Outline

Bellman Equation

$$v_{t,\Pi}(s) = \sum_a \pi_t(a|s) \sum_{s',r} p(s', r|s, a) \left(r + \gamma v_{t+1,\Pi}(s')\right)$$
$$= \sum_a \pi_t(a|s) r(s, a) + \gamma \sum_{s'} \sum_a p(s'|s, a) \pi_t(a|s) v_{t+1,\Pi}(s')$$



Bellman Equation

- Link between $v_{t,\Pi}$ and $v_{t+1,\Pi}$.
- Straightforward consequence of

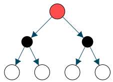
$$G_t = \sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} R_{t'} = R_{t+1} + \gamma \sum_{t'=t+2}^{T} \gamma^{t'-(t+2)} R_{t'} = R_{t+1} + \gamma G_{t+1}$$

and thus

$$\mathbb{E}[G_t|S_t = s] = \mathbb{E}[R_{t+1}|S_t = s] + \gamma \mathbb{E}[\mathbb{E}[G_{t+1}|S_{t+1}]|S_t = s]$$

Bellman Operator

$$\mathcal{T}^{\pi_t} : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}$$



$$\mathcal{T}^{\pi_t} v(s) = \underbrace{\sum_a \pi_t(a|s) r(s,a)}_{r_{\pi_t}(s)} + \gamma \sum_{s'} \underbrace{p(s'|s,a) \sum_a \pi_t(a|s)}_{P^{\pi_t}(s,s')} v(s')$$

Bellman Operator

- Affine operator from the space of state value functions to the space of state value functions.
- By construction,

$$v_{t,\Pi} = \mathcal{T}^{\pi_t} v_{t+1,\Pi}$$

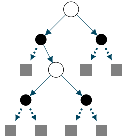
- r_{π_t} is the vector of average immediate rewards using policy π_t while P^{π_t} is the one step state transition matrix using policy π_t.

Outline

Finite Horizon: Naive Approach

$$v^T_{t,\Pi}(s) = \sum_{a_t, r_{t+1}, s_{t+1}, \cdots, r_T} \left(\sum_{t'=t+1}^{T} r_{t'} \right) \mathbb{P}_\Pi(A_t = a_t \ldots, R_T = r_T | S_t = s)$$

$$= \sum_{a_t, r_{t+1}, s_{t+1}, \cdots, r_T} \left(\sum_{t'=t+1}^{T} r_{t'} \right) \pi_t(a_t|s) \times \cdots \times p(s_T, r_T | s_{T-1}, a_{T-1})$$



Finite Horizon: Naive Approach

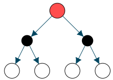
- Exhaustive exploration of the trajectories.
- Complexity of order $(|\mathcal{A}| \times |\mathcal{S}| \times |\mathcal{R}|)^{T-t}$ for the value function at time t.

- Complexity can be reduced to $(|\mathcal{A}| \times |\mathcal{S}|)^{T-t}$ by noticing that

$$v^T_{t,\Pi}(s) = \sum_{a_t, s_{t+1}, \cdots, s_{t-1}, a_{t-1}} \left(\sum_{t'=t+1}^{T} r(s_t, a_t) \right) \pi_t(a_t|s) \times \cdots \times p(s_T | s_{T-1}, a_{T-1})$$

Finite Horizon: Recursive Prediction

$$v^T_{T,\Pi} = 0$$
$$v^T_{t-1,\Pi} = \mathcal{T}^{\pi_{t-1}} v^T_{t,\Pi}$$



Finite Horizon: Recursive Prediction

- After time T, the finite horizon return $G^T_t = 0$ hence $v^T_{T,\Pi} = 0$ whatever the policy.
- The Bellman equation yields second equation.
- Equivalent rewriting
$$v^T_{t-1,\Pi}(s) = r_{\pi_{t-1}}(s) + \sum_{s'} P_{\pi_{t-1}}(s,s') v^T_t$$
- Complexity of order only $T \times |\mathcal{S}|^2(|\mathcal{A}| + |\mathcal{S}|)$ to compute all the value functions.

Finite Horizon: Value Iteration

Finite Horizon: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$ and policy Π
parameter: Horizon T
init: $v_T^T(s) = 0 \, \forall \, s \in \mathcal{S}, \; t = T$
repeat
  $t \leftarrow t - 1$
  for $\forall \, s \in \mathcal{S}$ **do**

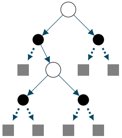
$$v_t^T(s) \leftarrow \sum_{a \in \mathcal{A}} \pi_t(a|s) \left(r(s,a) + \sum_{s' \in \mathcal{S}} p(s'|s,a) v_{t+1}^T(s') \right)$$

  end
until $t = 0$
output: Value functions v_t^T

- Most classical formulation

Discounted: Naive Approach

$$v^{\gamma}_{t,\Pi}(s) = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} \mathbb{E}_{\Pi}[R_{t'}|S_t = s] \simeq \sum_{t'=t+1}^{T} \gamma^{t} \mathbb{E}_{\Pi}[R_{t'}|S_t = s] = v^{\gamma,T}_{t,\Pi}(s)$$



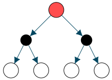
$$v^{\gamma,T}_{t,\Pi}(s) = \sum_{a_t, s_{t+1}, \cdots, s_{t-1}, a_{t-1}} \left(\sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} r(s_t, a_t) \right) \pi_t(a_t|s) \times \cdots \times p(s_T|s_{t-1}, a_{t-1})$$

Naive approach

- Exhaustive exploration of truncated trajectories.
- Back to the finite horizon setting...
- **Prop:** Control on the error as $\left| v^{\gamma}_{\Pi} - v^{\gamma,T}_{t,\Pi} \right|_{\infty} \leq \frac{\gamma^{T+1-t}}{1-\gamma} \max_{r \in \mathcal{R}} |r|$
- Relation between the error $\epsilon \simeq \gamma^{T-t}$ and the numerical complexity $C = (|\mathcal{A}| \times |\mathcal{S}|)^{T-t}$ of order $C \simeq \epsilon^{-1}$.

Discounted: Recursive Prediction with Naive Initialization

$$v^{\gamma}_{T,\Pi} \simeq v^{\gamma,T'}_{T,\Pi} = \tilde{v}_{T,\Pi}$$
$$v^{\gamma}_{t-1,\Pi} = \mathcal{T}^{\pi_{t-1}} v^{\gamma}_{t,\Pi} \simeq \tilde{v}_{t-1,\Pi} = \mathcal{T}^{\pi_{t-1}} \tilde{v}_{t,\Pi}$$

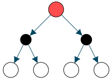


Recursive Prediction

- Requires an initialization at time T with a horizon T'.
- The Bellman equation yields the second equation.
- Complexity of order only $T \times |\mathcal{S}|^2(|\mathcal{A}| + |\mathcal{S}|)$ to compute all the value functions after the initialization of cost $(|\mathcal{A}| \times |\mathcal{S}|)^{T'-T}$.
- **Prop:** If the approximation error between $v^{\gamma}_{T,\Pi}$ and $v^{\gamma,T'}_{T,\Pi}$ is bounded by ϵ then
$$\|v^{\gamma}_{t,\Pi} - \tilde{v}_{t,\Pi}\|_\infty \leq \gamma^{T-t}\epsilon, \quad \forall t \leq T$$

Discounted and stationary: Bellman Equation

$$v_\Pi = \mathcal{T}^\pi v_\Pi$$
$$v_\Pi(s) = \sum_a \pi(a|s) r(s,a) + \gamma \sum_{s'} \sum_a p(s'|s,a) \pi(a|s) v_\Pi(s')$$

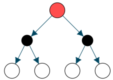


Bellman Equation

- Time independent value function v_Π.
- **Prop:** Unique solution of the linear equation $v_\Pi = \mathcal{T}^\pi v_\Pi$
- Complexity of order $(|A| + |S|) \times |S|^2$ to obtain the solution.

Discounted and stationary: Recursive Implementation

$$v_\Pi = \mathcal{T}^\pi v_\Pi$$
$$v_{k+1} = \mathcal{T}^\pi v_k \quad \text{with arbitrary } v_0$$



Bellman Iteration

- **Prop:** Unique fixed point of the Bellman operator $v \mapsto \mathcal{T}^\pi v$.
- **Prop:** The iterates $v_{k+1} = \mathcal{T}^\pi v_k$ converges toward v_Π and
$$\|v_k - v_\Pi\|_\infty \leq \gamma^k \|v_0 - v_\Pi\|_\infty$$
- Complexity of order $(k + |A|)|S|^2$ to obtain the kth iterate.
- Exponential decay of the error with respect to the complexity.

Bellman Operator and Contraction

$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty \leq \gamma \|v - v'\|_\infty$$

Proof

- By definition
$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty = \gamma \|P^\pi (v - v')\|_\infty$$
- It suffices then to notice that P^π is a transition matrix, so that
$$\sum_j P^\pi_{i,j} = 1$$
and thus $|\sum_j P^\pi_{i,j} z_j| \leq \max |z_j|$

Consequences

- Unicity of the solution of $\mathcal{T}^\pi v = v$.
- Linear decay γ^k of the error with the iterates.

Bellman Operator and Bellman Equation Solution

$$v_\Pi = \left(\sum_{k=0}^{\infty} \gamma^k \left(P^\pi\right)^k \right) r_\pi$$

A Closed Formula for the State Value Function

- $v_\Pi = \mathcal{T}^\pi v_\Pi \Leftrightarrow \left(I - \gamma P^\pi\right) v_\Pi = r_\pi$
- As P^π is a transition matrix, its eigenvalues are smaller than 1 and thus $\left(I - \gamma P^\pi\right)$ is invertible of inverse

$$\left(I - \gamma P^\pi\right)^{-1} = \sum_{k=0}^{\infty} \gamma^k \left(P^\pi\right)^k$$

- Could have been obtained without the Bellman equation as the $\left(\left(P^\pi\right)^k\right)_{s,s'}$ is, by construction, the probability of being at state s' at time k starting from s at time 0 and following Π.

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, discount factor γ, and stationary policy π
init: $\tilde{v}(s)\,\forall\, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  for $s \in \mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \sum_{a\in\mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s') \right)$$

  end
output: Value function $\tilde{v}$

- When to stop?

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, discount factor γ, and stationary policy π
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s)\,\forall\, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\mathrm{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \sum_{a\in\mathcal{A}} \pi(a|s)\left(r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\mathrm{prev}}(s')\right)$$

    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\mathrm{prev}}(s)|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

- **Prop:** when the algorithms stops

$$\|\tilde{v} - v_\Pi\|_\infty \leq \frac{\gamma}{1-\gamma}\delta$$

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration - Gauss-Seidel Version

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, discount factor γ, and stationary policy π
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s) \, \forall \, s \in \mathcal{S}$
repeat
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**
    $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}(s)$
    $$\tilde{v}(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) \tilde{v}(s') \right)$$
    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

- Gauss-Seidel variation mostly used in practice.
- No need to store the previous value function.

Outline

Optimal Policy

Optimal Policy

- An optimal policy $\Pi_\star$ should be better than any other policies:

$$\forall s, \forall t, v_{t,\Pi_\star}(s) = \sup_{\Pi} v_{t,\Pi}(s)$$

Several Questions

- Do this policy exists?
- Is it unique?
- How to characterize it?
- How to obtain it?

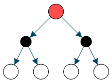
- Even the sup above could be an issue if it is not attained!

Finite Horizon and Optimal Policy

Explicit Recursive Solution

- After horizon T, any policy leads to a 0 return.
- At time $T-1$,
  - the total return G_T is the immediate return at time T and thus
  $$v_{T,\Pi^\star}(s) = \sup_{\pi(a|s)} \sum_a \pi(a|s) r(a,s) = \sup_a r(a,s)$$
  - the optimal policy $\pi^\star_{T-1}$ exists and is determistic.
- By recursion,
  - the total return at time $t-1$ is the immediate return at time t plus the total return at time $t-1$ and thus
  $$\begin{aligned} v_{t-1,\Pi^\star}(s) &= \sup_{\pi(a|s)} \sum_a \pi(a|s) \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{t,\Pi^\star}(s') \right) \\ &= \sup_a \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{t,\Pi^\star}(s') \right) \end{aligned}$$
  - the optimal policy $\pi^\star_{t-1}$ exists and is determistic.

Discounted Setting and Optimal Stationary Policy



Heuristic

- Optimal policy: $v_{\Pi^\star}(s) = \sup_\pi v_\Pi(s)$
- Stationary solution:

$$
\begin{aligned}
v_{\Pi^\star}(s) &= \sup_{\pi} \left(\mathcal{T}^\pi v_{\Pi^\star}\right)(s) \\
&= \sup_{\pi_t(\cdots|s)} \sum_a \pi(a|s) \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s') \right) \\
&= \sup_a \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s') \right)
\end{aligned}
$$

- Optimal deterministic policy: $\pi^\star(s) \in \operatorname{argmax}\left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s')\right)$.

- Is everything well defined? Yes but one has to be more cautious!

Optimal Value Function and Bellman Operator

Optimal Value Function

- Optimal value function: $v_\star(s) = \sup_\Pi v_\Pi(s)$
- Defined state by state so that it is not necessarily attained by a single $\Pi^\star$

Optimal Bellman operator

- Similar to the Bellman operator but do not depend on a policy:

$$\mathcal{T}^\star v(s) = \sup_a \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v(s') \right)$$
$$= \sup_\pi \sum_a \pi(a) \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v(s') \right)$$

Link between the two

- $v \geq \mathcal{T}^\star v$ implies $v \geq v_\star$.
- $v \leq \mathcal{T}^\star v$ implies $v \leq v_\star$.

Optimal Value Function and Bellman Operator

Bellman Operator and Fixed Point

- **Prop:** $\mathcal{T}^\star$ is a γ-contraction for the sup-norm and thus it exists a unique $v_{\star\star}$ such that $v_{\star\star} = \mathcal{T}^\star v_{\star\star}$.

Fixed Point and Optimal Value Function

- **Prop:** : $v_\star = v_{\star\star}$ and is thus the unique fixed point of $\mathcal{T}^\star$.
- **Proof:** $v_{\star\star} = \mathcal{T}^\star v_{\star\star}$ and thus $v_{\star\star} = v_\star$ according the link between the optimal value function and the Bellman operator.

- Does this mean something about policies?

Optimal Policy and Bellman Operator

Bellman Operator and Policy

- **Prop:** For any v, any policy π_v satisfying

$$\pi_v(s) \in \operatorname*{argmax}_a \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v(s') \right)$$

is such that $\mathcal{T}^\star v(s) = \sup_\pi \mathcal{T}^\pi v(s) = \mathcal{T}^{\pi_v} v(s)$

Bellman Operator and Optimal Policy

- **Prop:** Any stationary policy $\pi_\star$ satisfying

$$\pi_\star(s) \in \operatorname*{argmax}_a \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v^\star(s') \right)$$

is optimal.
- **Proof:** Indeed by construction, $\mathcal{T}^\star v_\star = \mathcal{T}^{\pi_\star} v_\star$ and thus, as $\mathcal{T}^\star v_\star = v_\star$, $v_{\pi_\star} = v_\star$.

Optimal Policy and Bellman Operator

Summary

- It exists a unique $v_\star$ such that $\mathcal{T}^\star v_\star = v_\star$
- $\forall s, v_\star(s) = \sup_\pi v_\pi(s)$
- Any policy $\pi_\star$ satisfying:

$$\forall s, \pi_\star(s) \in \operatorname*{argmax}_a \left(r(a,s) + \gamma \sum_{s'} p(s'|s,a) v^\star(s') \right)$$

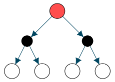
is optimal as $\forall s, v_{\pi_\star}(s) = v_\star(s) = \sup_\pi v_\pi(s)$

- Existence result but not (yet) a constructive algorithm!

Outline

Linear System and Linear Programming

$$v_\pi = \mathcal{T}^\pi v_\pi \qquad v_\star = \mathcal{T}^\star v_\star$$



Explicit Resolution of the Equations?

- Prediction:
  - Simple linear system for v_π.
  - Already mentionned before...
  - Complexity of order $(|A| + |S|)|S|^2$.
- Planning:
  - More complex linear programming system for $v_\star$ due to the max operator.
  - Optimal policy easily deduced from $v_\star$.
  - Complexity of order $(|A||S|)^3$.

Linear Programming

$$\text{From } \forall s, v(s) = \sup_a r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

$$\text{to } \min_v \sum_s \mu(s) v(s)$$

$$\text{such that } \forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

Different formulations but same solution

- Using $v \geq \mathcal{T}^\star v \Leftrightarrow v \geq v_\star$, the condition implies $v \geq v_\star$
- Now for any μ satisfying $\mu(s) > 0$, $\sum_s \mu(s) v(s) \geq \sum_s \mu(s) v_\star(s)$ as soon as the condition is satisfied, hence $v_\star$ is a solution.
- If for any state $v(s) > v_\star(s)$ then $\sum_s \mu(s) v(s) > \sum_s \mu(s) v_\star(s)$ and thus $v_\star$ is the unique minimizer.

Primal Problem

$$\text{Primal:} \quad \min_v \sum_s \mu(s) v(s)$$
$$\text{such that } \forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

Some properties

- Can be solved with a linear programming solver.
- Unicity of solution (and thus independence with respect to μ) can be proved without using $v_\star$.
  - **Proof:** let v_1 a solution for μ_1 and v_2 a solution for μ_2 then $\min(v_1, v_2)$ satifies the constraints. Furthermore if exists $v_2(s) < v_1(s)$ then $\min(v_1, v_2)$ is a strictly better solution for μ_2 which is impossible.

Dual Problem

Primal: $$\min_v \sum_s \mu(s) v(s)$$

$$\text{such that } \forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

Dual: $$\max_{\lambda(s,a)\geq 0} \sum_{s,a} \lambda(s,a) r(s,a)$$

$$\text{such that } \forall s, \sum_a \lambda(s,a) = \mu(s) + \gamma \sum_{s',a} p(s|s',a) \lambda(s',a)$$

Derivation

- Usual derivation through the Lagrangian:

$$\mathcal{L}(v,\lambda) = \sum_s \mu(s) v(s) + \sum_{s,a} \lambda(s,a) \left(r(s,a) + \gamma \sum_{s',a} p(s|s',a) v(s') - v(s) \right)$$

- Strong duality as Slater condition holds when $\gamma < 1$ with $v = \frac{1+\epsilon}{1-\gamma} \max_{s,a} r(s,a)$.

Dual and Interpretation

$$\text{Dual:} \quad \max_{\lambda(s,a)\geq 0} \sum_{s,a} \lambda(s,a) r(s,a)$$

$$\text{such that } \forall s, \sum_a \lambda(s,a) = \mu(s) + \gamma \sum_{s',a} p(s|s',a)\lambda(s',a)$$

$$\text{Interpretation}: \max_{\pi} \sum_{k=0}^{\infty} \gamma^k \sum_{s,a} \mathbb{P}(S_t = a, A_t = a | S_0 \sim \mu, \pi) \, r(s,a)$$

Interpretation in terms of policy

- For any feasible λ, define $u(s) = \sum_a \lambda(s,a)$ and the policy $\pi(a|s) = \lambda(s,a)/u(s)$.
- **Prop:** $u = (\mathrm{Id} - \gamma P^\pi)\mu = \sum_{k=0}^{\infty} \gamma^k (P^\pi)^k \mu$.
- **Prop:** $\lambda(s,a) = \pi(a|s)u(s) = \sum_{k=0}^{\infty} \gamma^k \mathbb{P}(S_t = a, A_t = a | S_0 \sim \mu, \pi)$
- Conversely for any π they is a feasible λ.
- Any optimal $\lambda_\star$ (and thus policy) satisfies $\lambda_\star(s,a) = 0$ if $v_\star(s) > r(s,a) + \gamma \sum_{s'} p(s'|s,a) v_\star(s')$ (optimal policy support)

Outline

Finite Horizon

Finite Horizon: Planning by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$
parameter: Horizon T
init: $v_T^T(s) = 0 \,\forall\, s \in \mathcal{S},\ t = T$
repeat
  $t \leftarrow t - 1$
  for $s \in \mathcal{S}$ **do**

$$v_t^T(s) \leftarrow \max_{a\in\mathcal{A}} \left(r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a) v_{t+1}^T(s') \right)$$

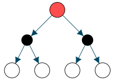
  end
until $t = 0$

output: Deterministic policy $\pi_t(s) \in \operatorname{argmax}_{a\in\mathcal{A}} \left(r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a) v_{t+1}^T(s') \right)$

- Algorithm used to prove the existence of an optimal policy.
- No necessarily unique as argmax may not be unique.

Optimal Value Function, Fixed Point and Contraction

$$v_\star = \mathcal{T}^\star v_\star \quad \text{and} \quad \|\mathcal{T}^\star v - \mathcal{T}^\star v'\|_\infty \le \gamma \|v - v'\|_\infty$$
$$\implies v_{k+1} = \mathcal{T}^\star v_k \to v_\star$$



Bellman Operator

- Properties of Optimal Bellman Operator:
  - $v_\star$ is a fixed point of $\mathcal{T}^\star$.
  - $\mathcal{T}^\star$ is a γ-contraction for the $\|\cdot\|_\infty$ norm.
- Classical fixed point theorem setting.
- Practical algorithm to approximate $v_\star$.

Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s) \,\forall\, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**
    $$\tilde{v}(s) \leftarrow \max_{a \in \mathcal{A}} r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')$$
    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

- Same convergence criterion (and similar proof) than in the planning case.
- Which policy?

Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s)\,\forall\, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \max_{a\in\mathcal{A}} r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')$$

$$\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$$

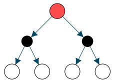
  end
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s) \in \operatorname{argmax}_a r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}(s')$

- Natural idea: define a policy using the argmax of the existence proof.
- Do we have a convergence guarantee on the resulting policy?

Value and argmax Policy

$$\tilde{\pi}(s) \in \operatorname*{argmax}_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)\tilde{v}(s')$$

$$\implies \|v_{\tilde{\pi}} - v_\star\|_\infty \leq \frac{2\gamma}{1-\gamma}\|\tilde{v} - v_\star\|_\infty$$



Value and argmax Policy

- Bound on the loss of the final policy!
- Rely on the fact that, by construction, $\mathcal{T}^{\tilde{\pi}}\tilde{v} = \mathcal{T}^\star\tilde{v}$
- **Proof:**

$$\begin{aligned}
\|v_{\tilde{\pi}} - v_\star\|_\infty &= \|\mathcal{T}^{\tilde{\pi}} v_{\tilde{\pi}} - \mathcal{T}^{\tilde{\pi}}\tilde{v} + \mathcal{T}^\star\tilde{v} - \mathcal{T}^\star v_\star\|_\infty \\
&\leq \|\mathcal{T}^{\tilde{\pi}} v_{\tilde{\pi}} - \mathcal{T}^{\tilde{\pi}}\tilde{v}\|_\infty + \|\mathcal{T}^\star\tilde{v} - \mathcal{T}^\star v_\star\|_\infty \\
&\leq \gamma\|v_{\tilde{\pi}} - \tilde{v}\|_\infty + \gamma\|\tilde{v} - v_\star\|_\infty \\
&\leq \gamma\|v_{\tilde{\pi}} - v_\star\|_\infty + 2\gamma\|\tilde{v} - v_\star\|_\infty
\end{aligned}$$

Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s)\,\forall\, s \in \mathcal{S}$
repeat
$\quad \tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
$\quad \Delta \leftarrow 0$
$\quad$ **for** $s \in \mathcal{S}$ **do**

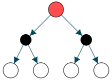
$$\tilde{v}(s) \leftarrow \max_{a\in\mathcal{A}} r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')$$

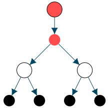
$$\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$$

$\quad$ **end**
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s) \in \operatorname{argmax}_a r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}(s')$

- **Prop:** $\|v_{\tilde{\pi}} - v_\star\|_\infty \leq \frac{2\gamma}{1-\gamma}\delta$

From State Value to State-Action Value Functions





$$v_\pi(s) = \mathbb{E}_\pi\left[\sum_k \gamma^k R_t | S_0 = s\right]$$

$$q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_k \gamma^k R_t | S_0 = s, A_0 = a\right]$$

$$\mathcal{T}^\pi v(s) = \sum_a \pi(a|s)\left(r(s,a) + \gamma \sum_{s'} p(s'|s,a)v(s')\right)$$

$$\mathcal{T}^\pi q(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) \sum_a \pi(a|s')q(s',a)$$

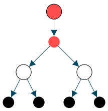
$$\mathcal{T}^\star v(s) = \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)v(s')$$

$$\mathcal{T}^\star q(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) \max_a q(s',a)$$

Two equivalent point of view?

- Everything could have been defined using the state-action point of view.
- Knowing v_π is equivalent to knowing q_π as
  $$v_\pi(s) = \sum_a \pi(a|s)q_\pi(s,a) \quad \text{and} \quad q_\pi(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a)v_\pi(s').$$

State-Action Bellman Operators



$$\mathcal{T}^{\pi} q(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) \sum_{a} \pi(a|s') q(s',a)$$

$$\mathcal{T}^{\star} q(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{a} q(s',a)$$

Properties

- **Prop:** $\mathcal{T}^{\pi}$ and $\mathcal{T}^{\star}$ are γ contractions for the $\|\cdot\|_\infty$ norm.
- **Prop:** q_π is the unique solution of $\mathcal{T}^{\pi} q = q$
- **Prop:** $q_\star$ defined $q_\star(s,a) = \sup_\pi q_\pi(s,a)$ is the unique solution of $q = \mathcal{T}^{\star} q$ and is attained for any policy $\pi_\star$ satisfying $\pi_\star(s) \in \operatorname{argmax} q_\star(s,a)$.
- **Prop:** Any such policy satisfies: $v_{\pi_\star}(s) = q_{\pi_\star}(s, \pi_\star(s)) = v_\star(s)$.

State-Action Value Iteration Algorithm

Discounted: Planning by State-Action Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{q}(s, a)\, \forall\, (s, a) \in \mathcal{S} \times \mathcal{A}$
repeat
  $\tilde{q}_{\text{prev}} \leftarrow \tilde{q}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**
    for $a \in \mathcal{A}$ **do**

$$\tilde{q}(s, a) \leftarrow \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) \max_{a'} \tilde{q}_{\text{prev}}(s', a') \right)$$

$$\Delta \leftarrow \max\left(\Delta, |\tilde{q}(s, a) - \tilde{q}_{\text{prev}}(s, a)|\right)$$

    end
  end
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s) \in \operatorname{argmax}_a \tilde{q}(s, a)$

- Same complexity but more storage than with state value function...
- but will be useful later!

Outline

Value Fonction vs Policy Point of View

$$v, q \longrightarrow \Pi \quad \text{or} \quad \Pi \longrightarrow v, q?$$

Planning

- Focus so far on value-fonction point of view!
- Heuristic: find a good approximation of the optimal value function and deduce a good policy.
- Can we work directly on the policy itself?

- For prediction, only the policy point of view makes sense!

Toward Policy Improvement

$$\forall s, \pi_+(s) \in \operatorname*{argmax}_a q_\pi(s, a) \Longrightarrow \forall v_{\pi_+}(s) \geq v_\pi(s)$$

Classical Policy Improvement Lemma

- **Prop:** Given a policy π and its q value-function, one can obtain a better policy with the argmax operator.
- **Prop:** If no improvement is possible, it means that π is already optimal.
- **Proof:** Use $\mathcal{T}^{\pi_+} v_\pi = \mathcal{T}^\star v_\pi \geq \mathcal{T}^\pi v_\pi = v_\pi$ to prove $(\mathcal{T}^{\pi_+})^k v_\pi \geq v_\pi$ which implies the result by letting k goes to $+\infty$.

- Leads to a sequential improvement algorith...

Policy Improvement Lemma

$$\mathbb{E}[v_{\pi'}(S_0)] - \mathbb{E}[v_\pi(S_0)] = \sum_{k=0}^{\infty} \gamma^k \mathbb{E}_{\pi'}\left[\sum_a \pi'(a|S_t)\left(q_\pi(S_t,a) - v_\pi(S_t)\right)\right]$$
$$= \sum_{k=0}^{\infty} \gamma^k \mathbb{E}_{\pi'}\left[\sum_a \left(\pi'(a|S_t) - \pi(a|S_t)\right) q_\pi(S_t,a)\right]$$

A Generic Improvement Lemma

- No assumptions on π and π'!
- Easy proof.
- Imply the previous lemma as $\max_a Q_\pi(s,a) - v_\pi(s) \geq 0$.
- Show that improvement choices are possible.

- Will prove to be useful later. . .

Policy Iteration

Discounted: Planning by Policy Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: Initial policy $\tilde{\pi}$
repeat
    Compute $q_{\tilde{\pi}}$.
    for $s \in \mathcal{S}$ **do**
        for $a \in \mathcal{A}$ **do**
            $\tilde{\pi}(s) \leftarrow \text{argmax}\, q_{\tilde{\pi}}(s, a)$
        end
    end
output: Deterministic policy $\tilde{\pi}$.

Some issues

- How to obtain q_π?
- When to stop?

Discounted: Planning by Policy Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: Initial policy $\tilde{\pi}$
repeat
  stable $\leftarrow 0$
  Compute $q_{\tilde{\pi}}$.
  for $s \in \mathcal{S}$ **do**
    $old - action \leftarrow \tilde{\pi}(s)$
    $\tilde{\pi}(s) \leftarrow \text{argmax}\, q_{\tilde{\pi}}(s, a)$
    if $\tilde{\pi}(s) \neq old - action$ **then**
      stable $\leftarrow 0$
    end
  end
until *stable* $=1$
output: Deterministic policy $\tilde{\pi}$.

Finite Setting

- Finite set of action-states implies a finite set of policy.
- Convergence of the algorithm in finite time!

Policy Iteration

Convergence Rate

- Crude analysis:
  - Bound after k steps of the algorithm
  $$\|v_{\pi_k} - v_\star\|_\infty \leq \gamma \|v_{\pi_{k-1}} - v_\star\|_\infty \leq \gamma^k \|v_{\pi_0} - v_\star\|_\infty$$
  $$\|v_{\pi_k} - v_\star\|_\infty \leq \frac{\gamma}{1-\gamma} \|v_{\pi_k} - v_{\pi_{k-1}}\|_\infty$$
  - Not much better than value iteration but much higher complexity as q_{π_k} is obtained by solving the Bellman equation!
- Much faster in practice...
- Clever analysis (Putterman):
  - Under some mild assumptions and provided $\|P^{\pi_k} - P^\star\| \leq K \|v_{\pi_k} - v_\star\|_\infty$ then
  $$\|v_{\pi_k} - v_\star\|_\infty \leq \frac{K\gamma}{1-\gamma} \|v_{\pi_{k-1}} - v_\star\|_\infty^2$$
  - May explain the better convergence in practice!

Outline

Value Iteration: (Relaxed) First Order Method

Value Iteration

- Iteration:
$$\begin{aligned} v_k &= \mathcal{T}^\star v_{k-1} \\ &= v_{k-1} + \left(\mathcal{T}^\star - \mathrm{Id}\right) v_{k-1} \end{aligned}$$
- Relaxation
$$v_k = v_{k-1} - \alpha \left(\mathrm{Id} - \mathcal{T}^\star\right) v_{k-1}$$
can be proved to converge for any $\alpha < \frac{2}{1+\gamma}$.
- Can be interpreted as a first order method with pseudo-gradient $\left(\mathcal{T}^\star - \mathrm{Id}\right) v_{k-1}$.
- No function corresponding to this gradient!

- Is there a better choice for α than $\alpha = 1$?
- No as the resulting operator is a contraction of constant
$$|1-\alpha| + \alpha\gamma \geq \gamma$$

Policy Iteration: Newton-Raphson Method

Policy Iteration

- Explicit iteration:

$$\text{Solve } v_{\pi_{k-1}} = \mathcal{T}^{\pi_{k-1}} v_{\pi_{k-1}}$$
$$\text{Let } \pi_k \text{ such that } \mathcal{T}^{\pi_k} v_{\pi_{k-1}} = \mathcal{T}^\star v_{\pi_{k-1}}$$

- Implicit iteration on v_{π_k}:

$$\begin{aligned} v_{\pi_k} &= (\mathrm{Id} - \gamma P^{\pi_k})^{-1} r_{\pi_k} \\ &= (\mathrm{Id} - \gamma P^{\pi_k})^{-1} \left(r_{\pi_k} + (\gamma P^{\pi_k} - \mathrm{Id}) v_{\pi_{k-1}} + (\mathrm{Id} - \gamma P^{\pi_k}) v_{\pi_{k-1}} \right) \\ &= v_{\pi_{k-1}} - (\mathrm{Id} - \gamma P^{\pi_k})^{-1} (\mathrm{Id} - \mathcal{T}^{\pi_k}) v_{\pi_{k-1}} \end{aligned}$$

- Can be interpreted as a second order method with pseudo-gradient $(\mathrm{Id} - \mathcal{T}^{\pi_k}) v_{\pi_{k-1}} = (\mathrm{Id} - \mathcal{T}^\star) v_{\pi_{k-1}}$ and pseudo-Hessian $(\mathrm{Id} - \gamma P^{\pi_k})$.

- Not a formal analysis but give a good insight on the better convergence of policy iteration.

Outline

Stability of Value and Policy Iteration

Ideal Value and Policy Iteration?

- Iterative algorithms.
- Convergence proofs assume perfect computation.
- What happens if we make a (small) error at each step?

- Particularly important for Policy Iteration in which one resolves a linear system at each step!

Value Iteration Stability

$$v_k = \mathcal{T}^\star v_{k-1} + \epsilon_{k-1}$$

$$\implies \|v_k - v_\star\|_\infty \leq \gamma^k \|v_0 - v_\star\|_\infty + \frac{\max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty}{1-\gamma}$$

$$\implies \|v_{\pi_k} - v_\star\|_\infty \leq \frac{2\gamma^{k+1}}{1-\gamma}\|v_0 - v_\star\|_\infty + \frac{2\gamma \max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty}{(1-\gamma)^2}$$

Stability with respect to approximations

- Proof relies on the contraction property of $\mathcal{T}^\star$ (hence similar results for $\mathcal{T}^\pi$).
- Error term $\frac{\max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty}{1-\gamma}$ can be replaced by $\sum_{k'=0}^{k-1} \gamma^{k-k'} \|\epsilon_{k'}\|_\infty$
- Convergence if $\|\epsilon_k\|_\infty$ tends to 0.
- Reach a neighborhood of the optimal solution if $\|\epsilon_k\|_\infty$ is bounded.

Policy Iteration

$$v_{k-1} = v_{\pi_{k-1}} + \epsilon_{k-1} \quad \text{and} \quad \mathcal{T}^{\pi_k} v_{k-1} = \mathcal{T}^\star v_{k-1} + \delta_{k-1}$$

$$\Rightarrow \|v_{\pi_k} - v_\star\|_\infty \leq \gamma^k \|v_{\pi_0} - v_\star\|_\infty + \frac{1}{(1-\gamma)^2}\left(2\gamma(2-\gamma) \max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty + \max_{0\leq k'<k} \|\delta_{k'}\|_\infty\right)$$

Stability with respect to approximations

- Quite involved proof but crude results.
- Error term $2\gamma(2-\gamma) \max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty + \max_{0\leq k'<k} \|\delta_{k'}\|_\infty$ can be replaced by

$$(1-\gamma)\sum_{k'=0}^{k-1} \gamma^{k-k'} \left(2\gamma(2-\gamma)\|\epsilon_{k'}\|_\infty + \|\delta_{k'}\|_\infty\right)$$

- Convergence if $\|\epsilon_k\|_\infty$ and $\|\delta_k\||_\infty$ tends to 0.
- Reach a neighborhood of the optimal solution if $\|\epsilon_k\|_\infty$ and $\|\delta_k\||_\infty$ are bounded.

- Justify why Policy Iteration only requires an approximate estimate of $v_{\pi_{k-1}}$, for instance obtained by Bellman iteration...

Outline

Modified Policy Iteration

Discounted: Planning by Generalized Policy Iteration

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: Initial q
repeat
  for $s \in \mathcal{S}$ **do**
    $\tilde{\pi}(s) \leftarrow \operatorname{argmax}_a q(s,a)$
  end
  repeat
    $q_{\text{prev}} \to q$
    for $(s,a) \in \mathcal{S}\times\mathcal{A}$ **do**
      $q(s,a) \leftarrow r(s,a) + \gamma \sum_{s,a'} p(s'|s,a)\tilde{\pi}(a'|s)q_{\text{prev}}(s,a)$
    end
output: Deterministic policy $\tilde{\pi}$.

- Algorithm driven by q.
- Flexibility in the number of prediction steps after each policy improvement steps.
- Special cases:
  - Large number: Policy Iteration with (small) error.
  - One: Value Iteration!

MPI Analysis

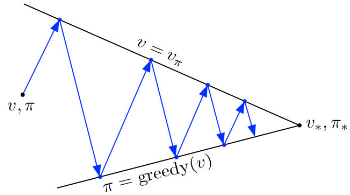
$$\mathcal{T}^{\pi_k} v_k = \mathcal{T}^\star v_k \quad \text{and} \quad v_{k+1} = \left(\mathcal{T}^{\pi_k}\right)^{m_k} v_k$$
$$\implies \|v_{k+1} - v_\star\|_\infty \leq \gamma \left(\frac{1-\gamma^{m_k}}{1-\gamma} \|P^{\pi_k} - P^\star\| + \gamma^{m_k} \right) \|v_k - v_\star\|_\infty$$

Convergence Results

- Quite technical proof.
- Valid only under the mild assumption $\mathcal{T}^\star v_0 \geq v_0$.
- Very fast decay provided $\|P^{\pi_k} - P^\star\|$ is small.

- No stability with arbitrary errors...
- Except if m_k is large enough (cf policy iteration).

Generalized Policy Iteration



General Policy Iteration

- Two simultaneous interacting processes:
  - One forcing the policy to correspond to the current value function (Policy Improvement)
  - One trying to male the current value function coherent with the current policy (Policy Evaluation)
- Several variations possible on the two processes.

- In GPI, the policy is driven by the value function.
- Typically, stabilizes only if one reaches the optimal value/policy pair.

Source: Sutton and Barto

State Update Order

Discounted: Prediction by Value Iteration - State Update Order

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, discount factor γ, and stationary policy π
init: $\tilde{v}(s)\,\forall\, s\in\mathcal{S}$
repeat

$\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$

for $s\in\mathcal{S}'\subset\mathcal{S}$ **do**

$$\tilde{v}(s)\leftarrow\sum_{a\in\mathcal{A}}\pi(a|s)\left(r(s,a)+\gamma\sum_{s'\in\mathcal{S}}p(s'|s,a)\tilde{v}_{\text{prev}}(s')\right)$$

end

output: Value function $\tilde{v}$

Classical strategies

- $\mathcal{S}'=\mathcal{S}$: classical iteration
- $\mathcal{S}'=\{s\}$: Gauss-Seidel
- $\mathcal{S}'=\{s,|\mathcal{T}^\pi\tilde{v}(s)-\tilde{v}(s)|>\epsilon\}$: Prioritized sweeping

- Converges provided all states are visited infinitely often. . .
- Gain in term of storage or focus on most interesting states. . .

Policy Improvement Variation

$$\text{Greedy}: \pi(\cdot|s) \in \operatorname{argmax}_a q(s,a) \iff \pi(\cdot|s) \in \operatorname{argmax}_{\tilde{\pi}} \sum_a \tilde{\pi}(a) q(s,a)$$

$$\text{Restricted}: \pi(\cdot|s) \in \operatorname{argmax}_{\tilde{\pi} \in \tilde{\Pi}_\epsilon} \sum_a \tilde{\pi}(a) q(s,a)$$

$$\text{Regularized}: \pi(\cdot|s) \in \operatorname{argmax}_{\tilde{\pi}} \sum_a \tilde{\pi}(a) q(s,a) + \epsilon P(\tilde{\pi})$$

Classical Variations

- ϵ-greedy: Restrict $\tilde{\pi}$ to the set of policy s.t. $\tilde{\pi}(a) \geq \epsilon/|\mathcal{A}|$
  - Explicit solution: $\pi(a|s) = \epsilon/|\mathcal{A}| + (1-\epsilon) \operatorname{argmax} q(s,a)$
  - Policy improvement property if ϵ decreases.
- Soft-max: Regularize by $\epsilon H(\tilde{\pi})$ where H is the entropy.
  - Explicit solution: $\pi(a|s) \propto \exp(q(s,a)/\epsilon)$
  - No classical policy improvement...

- Tends to greedy when ϵ goes to 0.
- Turn out to be interesting later...

Outline

Episodic Setting

$$\mathbb{E}_\pi\left[\min_t\{t, S_t = s_{abs}\}\right] < H \Rightarrow \|\mathcal{T}v - \mathcal{T}v'\|_\xi \leq \frac{H-1}{H}\|v - v'\|_\xi$$

Proper Policy

- A policy π is said to be H-proper if $\mathbb{E}_\pi\left[\min_t\{t, S_t = s_{abs}\}\right] \leq H < \infty$
- $\Rightarrow$ average duration of an episode using this policy less than a finite horizon H!

Bellman operators

- If a policy π is H-proper, the Bellman operator $\mathcal{T}^\pi$ is a $(H-1)/H$- contraction for a weighted sup-norm.
- If all the policies are H-propers, the optimal Bellman operator $\mathcal{T}^\star$ is a $(H-1)/H$-contraction for a weighted sup-norm.

- Under those strong assumptions, episodic setting $\simeq$ discounted setting with $\gamma = (H-1)/H$.
- Some results can be obtained under the much milder assumption that there is one proper policy and that any non-proper policy has at least one state for which $v_\pi(s) = -\infty$.

Episodic Setting and Discount

$$\exists H < \infty, \forall s, \mathbb{E}_\pi \left[\min_t \{ t, S_t = s_{\text{abs}} \middle| S_0 = s \} \right] < H$$
$$\iff \exists T, \gamma_T < 1, \forall s, \mathbb{P}_\pi (S_T = s_{\text{abs}} | S_0 = s) \geq 1 - \gamma_T$$

Episodic Setting and Discount

- Discounted setting: $\forall s, \mathbb{P}_\pi (S_T = s_{\text{abs}} | S_0 = s) = 1 - \gamma$
- Episodic setting: Generalization in which more states are needed to reach the absorbing state.
- **Prop:**
  - $H < \infty \implies \gamma_{(1+\epsilon)H} \leq \frac{1}{1+\epsilon}$
  - $\gamma_T < 1 \implies H < \frac{T}{1-\gamma_T}$

- Bertsekas equivalent assumption:

$$\exists \gamma_{|\mathcal{S}|} < 1, \forall s, \mathbb{P}_\pi \left(S_{|\mathcal{S}|} = s_{\text{abs}} \middle| S_0 = s \right) \geq 1 - \gamma_{|\mathcal{S}|}$$

Infinite Setting

- No issue with the rewards, as only the expectation is used.
- All the theory remains valid if the states are countable, but there is an issue in the algorithms, as we need to store/update an infinite number of states.
- The proof of existence of an optimal policy requires the max to be attained, which cannot be ensured in an infinite (even countable setting).

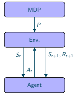
Some results...

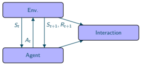
- **Thm:** If S is countable, there exists an ϵ-optimal (stationary) policy for any $\epsilon > 0$.
- **Thm:** If S is a Polish space (separable completely metrizable topological space),
  - there exists a (P, ϵ)-optimal (stationary) policy for any $\epsilon > 0$.
  - if each A_s is countable, there exists an ϵ-optimal (stationary) policy for any $\epsilon > 0$.
  - if each A_s is finite, there exists an optimal (stationary) policy.
  - if each A_s is a compact metric space, $r(s, a)$ is a bounded u.s.c. function on A_s and $p(B|s, a)$ is continuous in a for each Borel subset B and any s, there exists an optimal (stationary) policy.

- **Mainly technical difficulties...**

Outline

Reinforcement Learning

From 

to 

From Probability to Statistics?

- What to do if one has no knowledge of the underlying MDP?
- Only information through interactions!
- Prediction? Planning?
- Focus on the discounted setting

Outline

Monte Carlo, i.e. Just Play!

- Most simple way to evaluate a policy.

Just Play Following Policy Π

- Play N episodes following the policy.
- During each episode, compute the (discounted) gain.
- Compute the average gain.

- What is computed?

Average Gain or Value Function

$$\mathbb{E}[G_0] \quad \text{vs} \quad v_{t,\Pi}(s) = \mathbb{E}[G_t | S_t = s]$$

Prediction as Value Function Evaluation

- Not the same goal.
- By construction,
$$\mathbb{E}[G_0] = \sum_s \mu_0(s) v_{0,\Pi}(s)$$
- Much easier to compute the average gain than the value function (even if we use a stationary policy)

- Average gain is nevertheless the most classical way to evaluate a policy (with a single number).
- Implicit episodic setting if we do not want to use approximated gain.

Average Gain Estimation

Episodic: Evaluation by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $V = 0$,$n = 0$
repeat
  $n \leftarrow n + 1$
  $t \leftarrow 0$
  $G \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    Pick action A_t according to $\pi(\cdot|S_t)$
    $G \rightarrow G + \gamma^t R_{t+1}$
    $t \leftarrow t + 1$
  until *episod ends at time* T
  $V \leftarrow V + G$
until $n == N$
$V \leftarrow V/N$
output: Average gain V

Monte Carlo Prediction

- How to estimate $v_{t,\Pi}$?

Just Play Following Policy Π

- Play N episodes following the policy.
- During episode, record S_t and R_t.
- After each episode, compute recursively for each time t the gain G_t.
- Estimate $v_{t,\Pi}(s)$ by the average G_t over all trajectories such that $S_t = s$

- **May require a lot of game to have a non empty set for each state s at each time t**

Monte Carlo Prediction

- How to estimate v_Π for a stationary policy?

Just Play Following Policy Π

- Play N episodes following the policy.
- During each episode, record S_t and R_t.
- After each episode, compute recursively for each time t the gain G_t.
- Estimate $v_\Pi(s)$ by the average over all trajectories of all G_t such that $S_t = s$, whatever t.

- The same state may be reached several time during a single episode...
- First-visit variant: Use only the first visit of s for each episode.

Monte Carlo Prediction

Episodic: Prediction by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $\forall s, V(s), n = 0, N(s) = 0$
repeat
  $n \leftarrow n + 1$
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    (If First-visit) $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    Record R_{t+1}, S_{t+1}
    $t \leftarrow t + 1$
  until *episod ends at time* T
  $G_{T+1} = 0$
  $t \rightarrow T + 1$
  repeat
    $t \leftarrow t - 1$
    Compute $G_t = R_{t+1} + \gamma G_{t+1}$
    (If First-visit) $V(S_t) = V(S_t) + G_t$
  until $t = 0$
until $n == N$
for $s \in \mathcal{S}$ **do**
  $V(s) \leftarrow V(s)/N(s)$
end
output: Value function V

Monte Carlo Prediction Analysis

First-Visit Variant Analysis

- Straightforward analysis as all the used values for a given state s are independent.
- Variance of order $1/N(s)$ where $N(s)$ is the number of episodes where s is visited.
- Convergence if the number of visits goes to ∞.
- Strong assumption is practice, as some states may not be visited by a given policy (if we cannot play on the initial state).

- Every-visit works. . . but not necessarily better!

Outline

Monte Carlo Planning

- Can we use a MC approach to find a good policy?

A First Attempt

- Estimate $v_\pi(s)$ by $V_\pi(s)$ using MC.
- Compute $Q_\pi(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) V_\pi(s)$
- Enhance the current policy by setting $\pi(s) = \operatorname{argmax}_a Q_\pi(s,a)$

- Inspired by the Operations Research results...
- But unusable as r and p are unknown!

Monte Carlo Planning

A Second Attempt

- Estimate $q_\pi(s,a)$ by $Q_\pi(s,a)$ using MC.
- Enhance the current policy by setting $\pi(s) = \operatorname{argmax}_a Q_\pi(s,a)$

- Requires that $N(s,a)$ the number of times that an episode contains the state s followed by action a goes to ∞.
- Impossible with a deterministic policy!

Monte Carlo Planning

Classical Exploratory Policies...

- Stochastic policies ensuring that any action can occurs at any state.
- ϵ-exploratory policy: use a determistic policy and replace it with a random action with probability ϵ.
- Gibbs policy: use a policy where $\pi(a|s) \propto e^{\lambda Q(a,s)} > 0$.

A Final Attempt

- Start from an exploratory policy.
- Estimate $q_\pi(s, a)$ by $Q_\pi(s, a)$ using MC.
- Enhance the current policy while remaining a exploratory policy.

- Last step is not straightforward...
- except for ϵ-deterministic policy for which the ϵ-exploratory policy with base policy $\pi(s) = \operatorname{argmax}_a Q_\pi(s, a)$ works.
- No convergence proof.

Outline

Advanced Implementation of Monte Carlo Prediction

$$V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t))$$

On-Line Monte Carlo

- Average for a given state can be updated each time we have the gain G_t for a state S_t.
- Just use $\alpha(N) = 1/N$ and increment $N(S_t)$.
- No need to record the values between episodes...

- We still need to wait until the end of each episode to compute G_t.
- Can we do better?

Advanced MC Prediction

Episodic: Prediction by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $\forall s, V(s), n = 0, N(s) = 0$
repeat
  $n \leftarrow n + 1$
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    (If First-visit) $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    Record R_{t+1}, S_{t+1}
    $t \leftarrow t + 1$
  until *episod ends at time* T
  $G_{T+1} = 0$
  $t \rightarrow T + 1$
  repeat
    $t \leftarrow t - 1$
    Compute $G_t = R_{t+1} + \gamma G_{t+1}$
    (If First-visit) $V(S_t) = V(S_t) + \frac{1}{N(S_t)}(G_t - V(S_t))$
  until $t = 0$
until $n == N$
output: Value function V

- We still need to wait until the end of each episode to compute G_t.
- Can we do better?

Prediction with Temporal Differencies

$$\text{From} \quad V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t))$$

$$\text{to} \quad V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t)) \underbrace{(R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t))}_{\delta_t}$$

Bootstrap Strategy

- Replace G_t by an instantaneous estimate $R_{t+1} + \gamma V_\pi(S_{t+1})$.
- Amounts to replace $\gamma R_{t+2} + \gamma^2 R_{t+2} + \dots$ by an approximation of its expectation given S_{t+1}: $v_\pi(S_{t+1})$.
- Bootstrap as we use the current estimate $V_\pi(S_{t+1})$ instead of the true value.
- $\delta_t = R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t)$ is called a temporal difference.

- No need to wait until the end of the episodes!
- Can be used in the discounted setting.

TD Prediction

Discounted: Prediction by TD

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, V(s), n = 0, N(s) = 0,\ t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\,(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$
  - $t \leftarrow t + 1$
- **until** *episod ends at time T' or $t' = T$*

until $t' = T$
output: Value function V

- **But does this work?**

Prediction with Temporal Differencies

$$\mathbb{E}[\delta_t|S_t] = \mathbb{E}[R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t)|S_t] = (\mathcal{T}^\pi - \mathrm{Id})\, V_\pi(S_t)$$

TD and Bellman Operator

- TD as an approximate Policy Iteration:

$$\mathbb{E}[V_\pi]\,(S_t) \leftarrow V_\pi + \alpha(N(S_t))\,(\mathcal{T}^\pi - \mathrm{Id})\, V_\pi(S_t)$$

- Proof of convergence of this algorithm to a zero of $\mathcal{T}^\pi - \mathrm{Id}$, i.e. the fixed point of $\mathcal{T}^\pi$!
- Proof requires a mild assumption of α (satisfied by $\alpha(N) = 1/N$) and the strong assumption that $N(s)$ goes to ∞.

- MC could be interpreted in a similar way (stochastic approximation) by noticing that $\mathbb{E}[G_t - V_\pi(S_t)|S_t] = v_\pi(S_t) - V_\pi(S_t)$.
- Often use with a constant α

MC vs TD

$$V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t))$$

or $$V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))\underbrace{(R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t))}_{\delta_t}$$

MC vs TD

- Both are based on stochastic approximation.
- Both converges (under similar assumptions) to the correct value function.
- TD does not require to wait until the end of the episode.
- No theorical difference in the speed of convergence but often TD is better...
- Solve different approximate problems when used with a finite set of episodes:
  - MC compute the empirical gain from any state.
  - TD compute the value function of the empirical Bellman operator (the one obtained by using the empirical transition probabilities)

- If V_π is kept constant during an episode

$$G_t - V_\pi(S_t) = \sum \gamma^{t'-t}\delta_t$$

Outline

Stochastic Approximation

$$\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\implies \theta_k \to \{\theta, H(\theta) = 0\}$$

Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\text{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - the algorithm converges if we replace h_k by H.

- Convergence toward a neighborhood if α is kept constant (as often in practice).
- Most famous example are probably Robbins-Monro and SGD.
- Proof quite technical in general.
- The convergence with H is easy to obtain for a contraction.

Stochastic Approximation and ODE

From $\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k)$ with $h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$

to $\dfrac{d\tilde{\theta}}{dt} = H(\tilde{\theta})$

ODE Approach

- General proof showing that the algorithm converges provided the ODE converges.
- Rely on the rewriting the equation
$$\frac{\theta_{k+1} - \theta_k}{\alpha_k} = h_k(\theta_k) = H(\theta_k) + \epsilon_k + \eta_k$$
- α_k can be interpreted as a time difference allowing to define a time $t_k = \sum_{t' \leq t} \alpha_k$.
- $\theta(t)$ is piecewise affine and defined through its derivative at time $t \in (t_k, t_{k+1})$.
- This piecewise function remains close to any solution of the ODE starting from θ_k for an arbitrary amount of time provided k is large enough.

- More general proofs based on martingale.

Asynchronous Update

$$\text{From} \quad \theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\text{to} \quad \forall i, \theta_{k+1}(i) = \theta_k(i) + \alpha_k(i) h_k(\theta_k)(i)$$

Asynchronous Update

- Componentwise action on θ.
- Not necessarily the same stepsize $\alpha_k(i)$ for all components.
- $\alpha_k(i) = 0$ is permitted!
- Previous results hold provided for every component i, $\sum_k \alpha_k(i) \to \infty$ and $\sum_k \alpha_k^2(i) < \infty$,

- Exact setting of TD approximation!

Outline

Planning with Temporal Differencies

A State Value Function Attempt

- $V_\star$ is the fixed point of $\mathcal{T}^\star$.
- Approximate it as the zero of $\mathcal{T}^\star - \mathrm{Id}$.
- By construction

$$\mathcal{T}^\star v(S_t) = \max_a \mathbb{E}[R_{T+1} + \gamma v(S_{t+1}) | S_t, a]$$

- Not an expectation!

A State-Action Value Function Attempt

- $q_\star$ is the fixed point of $\mathcal{T}^\star$.
- Approximate it as the zero of $\mathcal{T}^\star - \mathrm{Id}$.
- By construction

$$\mathcal{T}^\star q(S_t, A_t) = \mathbb{E}\left[R_{t+1} + \gamma \max_a q(S_{t+1}, a) \middle| S_t, A_t\right]$$

- An expectation!

Discounted: Planning by Q-Learning

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    $$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\right)$$
    $t \leftarrow t + 1$
    $t' \leftarrow t' + 1$
  until *episod ends at time* T' *or* $t' = T$
until $t' = T$
output: Deterministic policy $\tilde{\pi}(s) = \text{argmax}_a Q(s,a)$

Planning with Q Learning

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)}_{\delta_t} \right)$$

Q-Learning

- Update is independent of the policy Π.
- Convergence of the Q-value function provided the policy is such that $N(s, a)$ tends to ∞ for any state and any action.
- Implies a convergence of the policy.
- Relies on temporal difference.

- Most classical (tabular) planning algorithm!

Outline

Planning with Policy Improvement

$$\text{from} \quad Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)}_{\delta_t} \right)$$

$$\text{to} \quad Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)}_{\delta_t} \right)$$

$$\Pi(S_t) = \operatorname{argmax}_a Q(S_t, a) (\text{plus exploration})$$

Policy Improvement

- More emphasis on the policy with a link between the policy used to play and the optimized policy.
- Almost equivalent to use the current policy in the Q-Learning algorithm.

SARSA

Discounted: Planning by SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
  $t \leftarrow 0$ Pick initial state S_0 following μ_0
  repeat
    $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    $Q(S_{t-1}, A_{t-1}) \leftarrow Q(S_{t-1}, A_{t-1}) + \alpha(N(S_{t-1}, A_{t-1}))\left(R_t + \gamma Q(S_t, A_t) - Q(S_{t-1}, A_{t-1})\right)$
    $\Pi(S_{t-1}) = \text{argmax}_a Q(S_{t-1}, a)$ (plus exploration)
    $t \leftarrow t + 1$
    $t' \leftarrow t' + 1$
  until *episod ends at time T' or $t' = T$*
until $t' = T$
output: Deterministic policy $\tilde{\pi}(s) = \text{argmax}_a Q(s, a)$

- Does this work?

SARSA and exploration

$$\Pi(S_t) = \underset{a}{\operatorname{argmax}}\, Q(S_t, a)(\text{plus exploration})$$

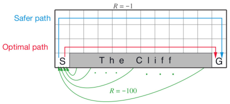
SARSA and Exploration

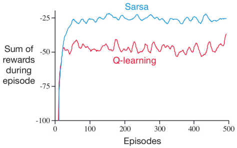
- No hope of convergence if we do not explore all possible actions (and states).
- Impossible if the policy used is deterministic.
- Exploration is required!
- Most classical choice: ϵ-greedy policy with a decaying ϵ.

- Convergence proof is harder than for Q-Learning.
- Relies on the similarity in the limit (when ϵ goes to 0) with the Q-Learning algorithm.

Outline

Q-Learning vs SARSA





How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

Exploration vs Exploitation

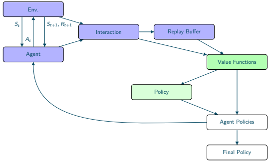
- Exploration: explore new policies to be able to discover the best ones.
- Exploitation: use good policies to obtain a good return.
- Exploration is a requirement.

- No tradeoff if we optimize only the final result!
- Tradeoff between the two if we consider that the returns during training matters!
- Q-learning use the first approach and SARSA try to tackle the second.
- Tradeoff if we study a regret:
$$\sum_t (\mathbb{E}_{\Pi_\star}[R_t] - \mathbb{E}_{\Pi_t}[R_t])$$
which forces us to be good as fast as possible.
- No natural definition in the discounted setting.

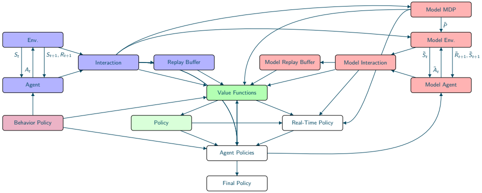
Outline

Advanced Tabular Reinforcemcent Learning

From



to



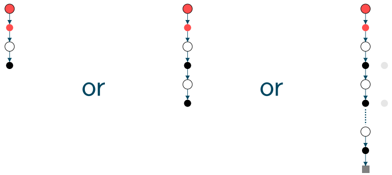
- Core idea: Approximate Bellman Operators with Stochastic Approximation...

Advanced Ideas?

- Between MC and TD?
- Off-policy vs on-policy?
- Exploration vs Exploitation?
- Model? Replay?
- Real-Time Planning?

Outline

n-steps



How many steps before backup?

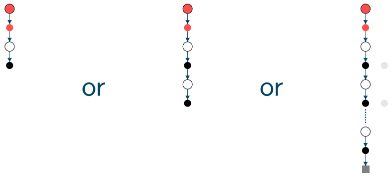
- One step: TD.
- As many steps as required to end the episod: MC.
- n-steps: n-steps TD.

$$\left(\mathcal{T}^{\Pi}\right)^n v(s) = \mathbb{E}_{\Pi}\left[\underbrace{R_{t+1} + \gamma R_{t+2} + \gamma^{n-1} R_{t+n} + \gamma^n v(S_{t+n})}_{G_{t:t+n}} \middle| S_t = s\right]$$

- Family of stochastic approximation algorithms:

$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\left(G_{t:t+n} - V(S_t)\right)$$

n-steps TD



$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\left(G_{t:t+n} - V(S_t)\right)$$

n-steps TD

- Convergence for prediction.
- Need to be combined with Policy Improvement for planning: n-steps SARSA.
- n-steps Q-learning could be an extension of API... but this means following the optimized policy Π... i.e. SARSA!

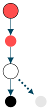
- Best convergence often for intermediate n.
- No proof beside TD for $n > 1$!

Discounted: Prediction by n-steps TD

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    $Q(S_{t-n}, A_{t-n}) \leftarrow Q(S_{t-n}, A_{t-n}) + \alpha(N(S_t, A_t))\left(G_{t-n:t} - Q(S_t, A_t)\right)$
    $t \leftarrow t + 1$
    $t' \leftarrow t' + 1$
  until *episod ends at time* T' *or* $t' = T$
until $t' = T$
output: State-Action value function Q

Expected SARSA

or



Expected SARSA

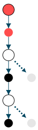
- The policy Π is known so that we can use it in a formula:
$$R_t + \gamma Q(S_t, A_t) \longrightarrow R_t + \gamma \sum_a \pi(a|S_t) Q(S_t, a)$$
- Make the update independent of the action chosen (and thus of the policy used to play).
- Reduce the variance for a computational cost.
- Amount to use the current estimate for $V(S_t)$...

Expected SARSA

Discounted: Prediction by Expected SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(R_{t+1} + \gamma \sum_a \pi(a|S_t) Q(S_{t+1}, a) - Q(S_t, A_t)\right)$
    $t \leftarrow t + 1$
    $t' \leftarrow t' + 1$
  until *episod ends at time* T' *or* $t' = T$
until $t = T$
output: State-Action value function Q

n-steps Tree Backup



n-steps Tree Backup

- At each time step, use the expected SARSA average over the action while replacing the Q value for the picked action by a deeper estimate.
  - 1-step return (Expected Sarsa)
$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$
  - 2-step return:
$$\begin{aligned} G_{t:t+2} = R_{t+1} &+ \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q_{t+1}(S_{t+1}, a) \\ &+ \gamma \pi(A_{t+1}|S_{t+1}) \left(R_{t+2} + \gamma \sum_a \pi(a|S_{t+2}) Q(S_{t+2}, a) \right) \\ = R_{t+1} &+ \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2} \end{aligned}$$

n-steps Tree Backup

- 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$

- 2-step return:

$$G_{t:t+2} = R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2}$$

$$= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+2} - Q(S_{t+1}, A_{t+1})\right)$$

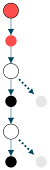
- Recursive definition of n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) \left(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1})\right)$$

- TD update

$$Q(S_{t-n}, A_{t-n}) = Q(S_{t-n}, A_{t-n}) + \alpha(N(S_{t-n}, Q_{t-n})) \left(G_{t-n:t} - Q(S_{t-n}, A_{t-n})\right)$$

$Q(\sigma)$

Between  and 

Sampling or Averaging

- Unifying algorithm!
- Recursive definition of n-step return:

$$G_{t:t+n} = R_{t+1} + \sigma G_{t+1:t+n} + (1-\sigma)\Big(\gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma\pi(A_{t+1}|S_{t+1})\,(G_{t+1:t+n} - Q(S_{t+1}, A_{t+1}))\Big)$$

λ-Return

Averaged n-steps return?

- n-step return:
$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$
- Averaged n-step return: (compound update)
$$G_t^{\omega} = \sum_{n=1}^{\infty} \omega_n G_{t:t+n} \quad \text{with} \sum_{i=n}^{\infty} \omega_n = 1$$
- TD(λ): specific averaging
$$\begin{aligned} G_t^{\lambda} &= (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n} \\ &= (1-\lambda) \sum_{n=1}^{T-t} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t} G_t \quad \text{(Episodic)} \end{aligned}$$
interpolating between TD (a.k.a TD(0)) and MC for $\lambda = 1$.
- Can be mixed with tree backup strategies (TB(λ))

λ-return and Temporality

True λ-return

- Require to wait until the end of an episode before we can update.
- Unusable in a non episodic setting!

Truncated λ-return

- Truncated λ-return:

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{H-t} \lambda^{n-1} G_{t:t+n} + \lambda^{H-t} G_{t:H}$$

- The virtual horizon H may vary during the algorithm.

λ-return and Temporality

Temporality

- *n*-step return

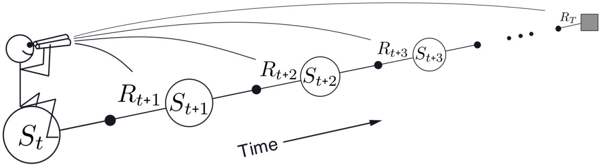
$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

  depends on a current estimate V (or Q)!
- In G_λ should we use
  - an estimate available at time t?
  - an estimate available at time $t+n$?
  - an estimate available at time H?
- Off-Line vs On-Line!
  - Off-line: keep V constant during the episodes.
  - On-line: Used updated V when available.
  - True on-line (Sutton and Barto): restart algorithm with a growing horizon.

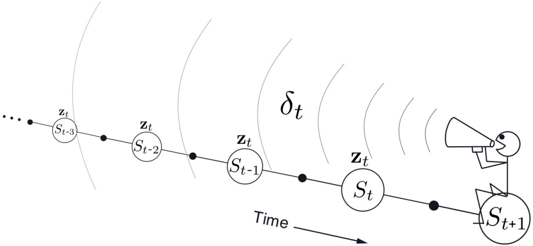
Outline

Forward and Backward Point of View

From a forward view



To a backward one:



Source: Sutton and Barto

Returns and Temporal Differencies

- n-step returns:

$$\begin{aligned} G_{t:t+n} - Q(S_t, A_t) &= R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} \\ &\quad + \gamma^n Q(S_{t+n}, A_{t+n}) - Q(S_t, A_t) \\ &= \sum_{l=1}^{n} \gamma^{l-1} (R_{t+l} + \gamma Q(S_{t+l}, A_{t+l}) - Q(S_{t+l-1}, A_{t+l-1})) \\ &= \sum_{l=0}^{n-1} \gamma^{l-1} \delta_{t+l} \end{aligned}$$

- λ return:

$$\begin{aligned} G_t^\lambda - Q(S_t, A_t) &= (1-\lambda) \sum_n \lambda^n (G_{t:t+n} - Q(S_t, A_t)) \\ &= \sum_{n=0} \lambda^n \gamma^n \delta_{t+n} \end{aligned}$$

Forward View and Backward View

Forward View

- Updates:

$$Q_t(s,a) = Q_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)} \alpha_t(s,a) \left(\sum_{t'' \geq t} \lambda^{t''-t} \gamma^{t''-t} \delta_{t''} \right)$$

- Cumulative updates:

$$Q_t(s,a) = Q_0(s,a) + \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right)$$

- Limit:

$$Q_\infty(s,a) = Q_0(s,a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right)$$

- Focus on the update place.

Forward View and Backward View

Limit(s)

- Limit:

$$\begin{aligned} Q_\infty(s,a) &= Q_0(s,a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \left(\sum_{t''\geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right) \\ &= Q_0(s,a) + \sum_{t''} \delta_{t''} \sum_{t'\leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'} \end{aligned}$$

- Focus on the update place or and the temporal differencies...

Forward View and Backward View

Backward View

- Same limit with cumulative udpates over temporal differencies
$$Q_t(s,a) = Q_0(s,a) + \sum_{t'' \leq t} \delta_{t''} \sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t''-t'} \gamma^{t''-t'}$$
- Updates
$$Q_t(s,a) = Q_{t-1}(s,a) + \delta_t \underbrace{\sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'}}_{z_t(s,a)}$$
- Pseudo Eligibility trace:
$$\begin{aligned} z_t(s,a) &= \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s,a) \lambda^{t-t'} \gamma^{t-t'} \\ &= \lambda \gamma z_{t-1}(s,a) + \alpha_t(s,a) \mathbf{1}_{(s,a)=(S_t,A_t)} \end{aligned}$$

- Proof of convergence toward the same target.

Eligibility Trace

$$Q_t(s,a) = Q_{t-1}(s,a) + \alpha_t \delta_t z_t(s,a)$$

Eligibility Trace

- Focus on temporal differencies with simultaneous update on all states.
- TD(λ) eligibility trace: $z_t(s,a) = \lambda\gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$
- Strictly equivalent to the previous scheme for constant stepsize
- Other eligibility trace:
  - Replacing trace:
    $$z_t(s,a) = \begin{cases} 1 & \text{if } (s,a) = (S_t,A_t) \\ \lambda\gamma z_{t-1}(s,a) & \text{otherwise} \end{cases}$$
  - Time dependent trace:
    $$z_t(s,a) = c_t\gamma z_{t-1}(s,a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$$
    where c_t is defined *in a appropriate way* to ensure the convergence of the algorithm.

- Need to store (and update) this information. . .

Temporal Differencies

$$\delta_t?$$

Temporal Differencies

- Basic temporal differencies:

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

- Expected temporal differencies:

$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$

- Average of both:

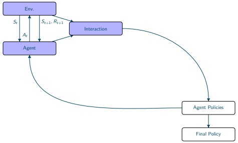
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$

- Only expected temporal average is independent of the next action.
- No generic proof of convergence. . .

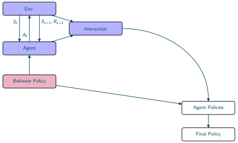
Outline

On-Policy vs Off-Policy

From



to



On-Policy vs Off-Policy

- On-Policy: the policy b used to interact is the same than the policy Π evaluated or optimized.
- Off-Policy: the policy b used to interact may be different from the policy Π evaluated or optimized.

- Off-Policy allows in particular to (re)use interactions from previous experiments.
- Q-learning was possible in off-policy setting.

Importance Sampling

$$\rho_{t:t'} = \frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}|S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}|S_t)} = \frac{\pi(A_t|S_t)\ldots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\ldots b(A_{t'}|S_{t'})}$$

Importance Sampling

- For any law p and q, and any function g
$$\mathbb{E}_p[g(x)] = \mathbb{E}_q\left[\frac{p(x)}{q(x)}g(x)\right]$$
provided $q(x) = 0$ implies $p(x) = 0$.
- $\mathbb{V}\text{ar}_q\left[\frac{p(x)}{q(x)}g(x)\right]$ may be large with respect to $\mathbb{V}\text{ar}_p\left[g(x)\right]$ if the ratio $p(x)/q(x)$ is large. . .

Importance Sampling for Trajectories

- For any trajectory $\tau_{t:t'} = S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1}),$,
$$\frac{\mathbb{P}_\Pi(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1})|S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \ldots, R_{t'}, S_{t'}, A_{t'}(, R_{t'+1}, S_{t'+1})|S_t)} = \frac{\pi(A_t|S_t)\ldots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\ldots b(A_{t'}|S_{t'})}$$

Importance Sampling and Returns

$$\mathbb{E}_\Pi[g(\tau_{t:t'})|S_t = s] = \mathbb{E}_b[\rho_{t:t'} g(\tau_{t:t'})|S_t = s] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\dots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\dots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t = s] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\rho_{t:(t'-1)} \left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} V(S_{t'})\right) \middle| S_t = s\right] \\
&= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \rho_{t:(t''-1)} \gamma^{t''-t-1} R_{t''} + \rho_{t:(t'-1)} \gamma^{t'-t} V(S_{t'}) \middle| S_t = s\right]
\end{aligned}$$

Importance Sampling and Returns

$$\mathbb{E}_\Pi[g(\tau_{t:t'})|S_t, A_t] = \mathbb{E}_b\left[\rho_{(t+1):t'} g(\tau_{t:t'})\middle|S_t, A_t\right] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t)\ldots\pi(A_{t'}|S_{t'})}{b(A_t|S_t)\ldots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}
\mathbb{E}_\pi[G_{t:t'}|S_t, A_t] &= \mathbb{E}_\pi\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\rho_{(t+1):(t'-1)}\left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\right)\middle|S_t, A_t\right] \\
&= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} \rho_{(t+1):(t''-1)} R_{t''} + \rho_{(t+1):t'} \gamma^{t'-t} Q(S_{t'}, A_{t'})\middle|S_t, A_t\right]
\end{aligned}$$

- No correction if $t' = t + 1$

λ-return

λ-return

- Recursive definition of the λ-return:

$$G_t^\lambda | S_t = R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) + \lambda G_{t+1}^\lambda \Big)$$

- Off-line correction

$$G_t^\lambda | S_t = \rho_{t:t} \left(R_{t+1} + \gamma \left((1-\lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right) \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \Big((1-\lambda) (\sigma Q(S_{t+1}, A'_{t+1}) + (1-\sigma) (\sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1}|S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right))) + \lambda \rho_{t+1:t+1} G_{t+1}^\lambda \Big)$$

where A'_{t+1} is drawn following π (or multiply by $\rho_{t+1:t+1}$ to use A_{t+1}).

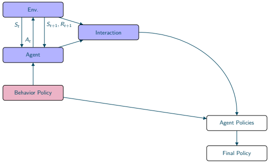
Temporal Differencies

$$\delta_t?$$

Temporal Differencies

- Basic temporal differencies:
$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A'_{t+1}) - Q(S_t, A_t)$$
with A'_{t+1} drawn using π.
- Expected temporal differencies:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$
without any correction.
- Average of both:
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma\sigma Q(S_{t+1}, A_{t+1}) + \gamma(1-\sigma) V(S_{t+1}) - Q(S_t, A_t)\\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma\sigma \left(Q(S_{t+1}, A'_{t+1}) - V(S_{t+1})\right) - Q(S_t, A_t)\end{aligned}$$
with A'_{t+1} drawn using π.

Off-Policy Algorithm



Off-Policy Correction

- Replace any estimate of the gain by an importance-sampling corrected one.
- Works well for prediction.
- Can be combined with policy improvement (a la SARSA) but less (no?) theoretical guarantees.

Retrace(λ)

$$\widetilde{\mathcal{T}}Q(s,a) = Q(s,a) + \mathbb{E}_b\left[\sum_{t\geq 0}\gamma^t\left(\prod_{t'=1}^{t} c_{t'}\right)\delta_t \middle| S_0 = s, A_0 = a\right]$$

$$c_t = c(A_t, S_t, A_{t-1}, S_{t-1}, \cdots, A_0, S_0)$$

$$\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[R_{t+1} + \gamma\mathbb{E}_\pi[Q(S_{t+1},\cdot)] - Q(S_t, A_t)|S_t, A_t]$$

Generic Off-Policy Algorithm

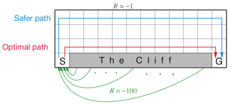
- Generic off-line algorithm including
  - Importance sampling: $c_t = \rho_{t:t} = \pi(A_t|S_t)/b(A_t|S_t)$
  - TB(λ): $c_t = \lambda\pi(A_t|S_t)$
  - Retrace(λ): $c_t = \lambda\min(1, \pi(A_t|S_t)/b(A_t/S_t))$
- **Prop:** Q_π is a fixed point as $\mathbb{E}_b[\delta_t|S_t, A_t] = \mathbb{E}[\mathcal{T}^\pi Q(S_t, A_t) - Q(S_t, A_t)|S_t, A_t]$.
- **Prop:** $\widetilde{\mathcal{T}}$ is a contraction provided $c_t \leq \rho_t = \pi(A_t|S_t)/b(A_t|S_t)$.
- Convergence for Importance sampling, TB(λ) and Retrace(λ) for any b.
- Partial results for policy improvement under more assumptions.

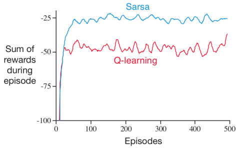
- For Q(λ), $c_t = \lambda$, convergence if $\|\pi(|s) - b(|s)\|_1 \leq \epsilon$ and $\lambda \leq (1-\gamma)/(\gamma\epsilon)$.

Discounted

Outline

Q-Learning vs SARSA





How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

Exploration vs Exploitation

- Exploration: explore new policies to be able to discover the best ones.
- Exploitation: use good policies to obtain a good return.
- Exploration is a requirement.

- No tradeoff if we optimize only the final result!
- Tradeoff between the two if we consider that the returns during training matters!
- Q-learning use the first approach and SARSA try to tackle the second.
- Tradeoff if we study a regret:
$$\sum_t \left(\mathbb{E}_{\Pi_\star}[R_t] - \mathbb{E}_{\Pi_t}[R_t]\right)$$
which forces us to be good as fast as possible.
- No natural definition in the discounted setting.

Bandits

$$\mathcal{S} = \{0\} \quad \text{and} \quad A = \{1, \dots K\} \quad \text{and} \quad r(s, a) = r_a$$

Bandits

- Very simple toy model where there is only one state!
- Optimal policy: pick $a_\star \in \operatorname{argmax} r_a$.
- Q estimation: estimate r_a by playing action a.
- Strategy:
  - Every arm has to be played until we are sure they are bad.
  - Best arm should be played as often as possible to maximime the rewards during the learnig phase.
- Simple enough setting to obtain result on the regret

$$r_T = \sum_{t \leq T} (r_{a_\star} - R_t)$$

- We will use $\Delta_a = r_{a_\star} - r_a$ and assume that $R|a$ is 1-subgaussian.

Explore Then Commit

Explore Then Commit (Random Exploration)

- Play the arm successively during Km steps and then play the optimal one during $T - Km$ steps.
- **Prop:**

$$\mathbb{E}[r_T] \leq \min(m, T/K) \sum_{a=1}^{k} \Delta(a) + \max(T - mK, 0) \sum_{a=1}^{K} \Delta(a) \exp(-m\Delta(a)^2/4)$$

Furthermore,

$$\mathbb{P}(a_T = a_*) \geq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)$$

- With $m \propto \log T$, logarithmic regret: $\mathbb{E}[r_T] \leq O(\log T)$ for
- but $\mathbb{E}[r_T] = O(T)$ for any fixed m.

ϵ-greedy Strategy

ϵ-greedy Strategy

- Estimate $Q(a) = r_a$ by MC:

$$Q_t(a) = \frac{\sum_{t'=1}^{t-1} \mathbf{1}_{A_{t'}=a} R_{t'}}{\sum_{i=1}^{t-1} \mathbf{1}_{A_{t'}=a}}$$

- Pick arm a at time t using

$$\pi_t(a) = \begin{cases} \epsilon_t/K + (1-\epsilon_t) & \text{if } a = \operatorname{argmax}_{a'} Q_t(a') \text{ (only the smallest if necessary)} \\ \epsilon_t/K & \text{otherwise} \end{cases}$$

- **Prop:**

$$\mathbb{E}[r_T] \geq \sum_{t=1}^{T} \frac{\epsilon_t}{K} \sum_{a=1}^{K} \Delta(a)$$

ϵ-greedy Strategy

ϵ-greedy Strategy

- **Prop:**

$$\mathbb{P}(A_T = a_*) \geq 1 - \epsilon_T - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4K)}$$

with $\Sigma_T = \sum_{s=1}^{T} \epsilon_s$.
Furthermore,

$$\mathbb{P}(a_* = \operatorname{argmax} Q_{T,a}) \geq 1 - \Sigma_t \exp(-\Sigma_T/(6K)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4K)}$$

If $\epsilon_t = c/t$,

$$\mathbb{E}[r_T] \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T) + 1}{K} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

as soon as $c/(6K) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4K < 1$.
If $\epsilon_t = c \log(t)/t$ then

$$\mathbb{E}[r_T] \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T) + 1)}{K} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

UCB Strategy

Upper Confidence Bound

- Use an optimistic strategy to pick the best arm

$$A_t = \operatorname{argmax} Q_t(a) + \sqrt{\frac{c \log t}{N_t(a)}}$$

- **Prop:**

$$\mathbb{E}[r_T] \leq C_c \sum_a \Delta(a) + \sum_a \frac{4c \ln T}{\Delta(a)}.$$

with $C_c < +\infty$ as soon as $c > 3/2$
Furthermore

$$\mathbb{P}(A_T = a_*) \geq 1 - 2KT^{-2c+2}$$

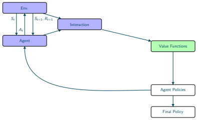
as soon as $T \geq \max_a \frac{4c \ln T}{\Delta(a)^2}$.

- Optimal regret!
- Hard to extend to RL setting but shows that ϵ-greedy may not be optimal.
- Bayesian approach possible: Thompson sampling.

Outline

Model Based Approach

From



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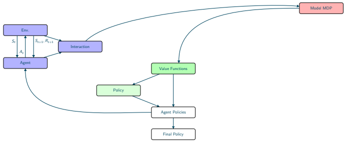

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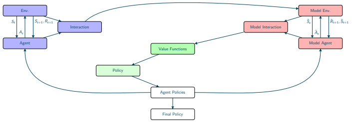
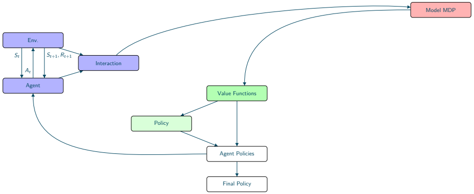


Model Based Approach

- Use the interactions to learn a model...
- that can be used to learn a good policy.
- This model can be:
  - a MDP,
  - a simulator.

- Often easier to obtain a simulator.

Model based and MDP

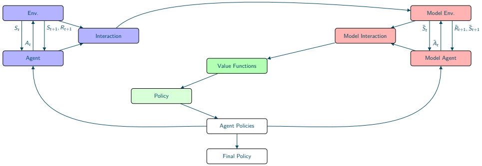


Estimated MDP: back to OR

- MDP can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated MDP, prediction and planning can be done using OR.

- Implicitely done by TD(0) when doing several passes.
- Model should be checked/improved as much as possible when new trajectories arrive.

Model based and RL

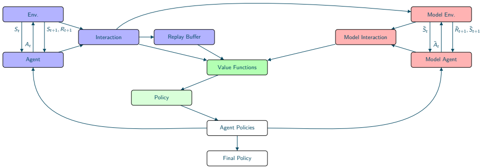


Estimated Simulator: back to RL

- Simulator can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated simulator, prediction and planning can be done using RL.

- Model should be checked/improved as much as possible when new trajectories arrive.

Model Free and Model Based Approach



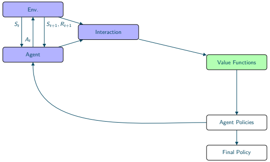
Dyna

- Combine true interactions with simulated ones.
- Simultaneous acting, model learning, OR learning and RL learning.
- Search for a tradeoff between the (slow) learning RL algorithm and the (wrong) model OR algorithm.
- Need to deal with schedule!

Outline

Replay Buffer and Prioritized Sweeping

From



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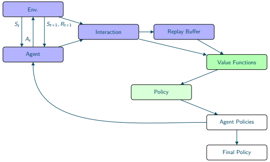
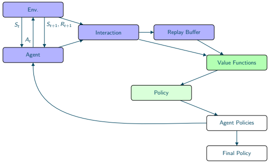


Replay Buffer and Prioritized Sweeping

- Can we reuse previous interactions?
- In which order?

Replay Buffer

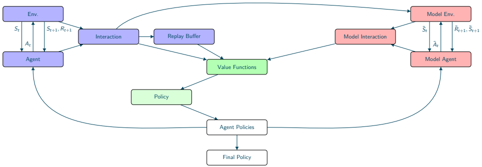


Replay Buffer

- Store previous interactions (trajectories) in a first-in first-out buffer.
- Draw a subsequence from those interactions (trajectories) and use it in a RL algorithm:
  - On-line: if the trajectory comes from the same policy.
  - Off-line: if the trajectory comes from a different policy.

- Similar to a simulator but no arbitrary choice of state or action.
- Often use with on-line algorithm if the policy has only mildy evolved...

Prioritized Sweeping



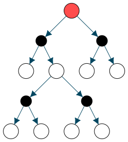
Prioritized Sweeping

- Plain Replay Buffer: subsequence drawn uniformly.
- Prioritized Sweeping: subsequence drawn favoring states with large temporal differencies.

- Can be combined with a model approach.

Outline

Real-Time Planning

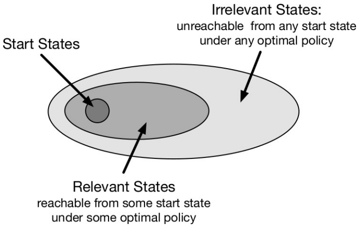


Real-Time Planning

- Can we optimize the policy at the current state?
- Do we need to optimize it everywhere?
- What is required?

- Planning at decision time. . .

Real-Time Dynamic Programming

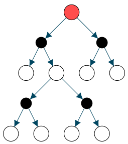


- Warmup in Dynamic Programming. . .

RT DP

- Use trajectories to sample the states to update.
- Convergence holds with exploratory policy.
- Optimal policy does not require to specify the action in irrelevant states.
- Convergence holds even without full exploration in some specific cases!

- In practice, seems to be computationaly efficient.



Planning At Decision Time

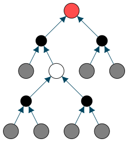
- Can we find a good action A_t at S_t... without having it precomputed?
- Policy Improvement

$$A_t = \text{argmax}\, Q_t(S_t, \cdot)$$

  can be seen as a first step.
- How to go deeper?

- **A model or a simulator will be required!**

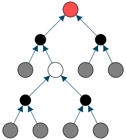
Heuristic Search



Heuristic Search

- Requires the knowledge of the MDP and of a heuristic based value function V.
- Strategy:
  - Build a limited depth tree by stopping after a few steps and at some specific states.
  - Backup the heuristic based value function using Dynamic Programming (Optimal Bellman operator).
  - Pick the action having the hight value.
- The deeper the better... but the more expensive due to branching!
- Requires a *suitable* heuristic...

Rollout Algorithm

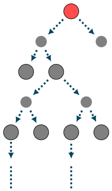


Rollout Policy

- Use a MC estimate with a default policy instead of a heuristic.
- Backup those estimates using Dynamic Programming.
- Simulation can even start after the first action (as in Policy Improvement).

- The values are (most of the time) discarded for the next state.

Monte Carlo Tree Search



Monte Carlo Tree Search

- Simultaneour tree growing, rollout and backup by DP.
- Repeat 4 steps:
  - Selection of a sequence of actions according to the current values with a tree policy.
  - Expansion of the tree at the last node without values.
  - Simulation with a rollout policy to estimate the values at this node.
  - Backup of the value by relaxed Dynamic Programming.
- MCTS focuses on promising paths using a UCB approach.

Monte Carlo Tree Search

Monte Carlo Tree Search

- Simultaneour tree growing, rollout and backup by DP.
- Repeat 4 steps:
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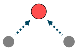
Monte Carlo Tree Search



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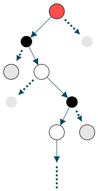

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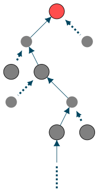

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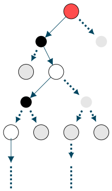

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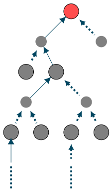

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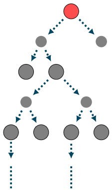

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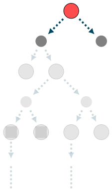

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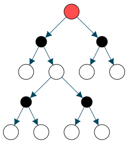
Model Predictive Control?



Model Predictive Control

- Open loop optimization:

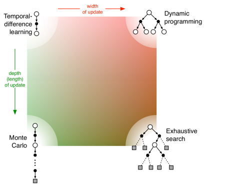
$$\max_{a_t, a_{t+1}, \dots, a_{t+h}} \mathbb{E}\left[\sum_{t'=t}^{t+h} R_t\right]$$

  using a predictive model (simulator).
- Do not take into account state uncertainties in the control choice...
- But much simpler optimization...
- and equivalence for a linear Gaussian model.

- Extensively used for short-term planning in Control.
- May be combined with value functions after $t + h$.

Outline

Approximation?



Tabular Setting

- Require to store the state(-action) values (a table).
- Requirement in both OR and RL.

Approximation!

- Use instead approximated value functions.
- What is a good approximation?
- How to use them?

- Focus on value-functions. . .

Outline

Approximated Value Functions

$$V(s) \Longrightarrow V_{\boldsymbol{w}}(s)$$
$$Q(s,a) \Longrightarrow Q_{\boldsymbol{w}}(s,a)$$

Parametric Model

- Reduce dimensionality by storing $\boldsymbol{w}$ instead of all the values.
- Linear: $V_{\boldsymbol{w}}(s) = \langle \Phi(s), \boldsymbol{w} \rangle$ and $Q_{\boldsymbol{w}}(s,a) = \langle \Phi(s,a), \boldsymbol{w} \rangle$
  - $\Phi(s)$ and $\Phi(s,a)$ are features associated to the states(-actions).
  - Tabular setting corresponds to $(\Phi)_{s'(,a')}(s(,a)) = \mathbf{1}_{s'=s(,a'=a)}$.
  - Often used in theoretical analysis.
- Deep Learning: $V_{\boldsymbol{w}}(s) = \mathrm{NN}_{\boldsymbol{w}}(\Phi(s))$ and $Q_{\boldsymbol{w}}(s,a) = \mathrm{NN}_{\boldsymbol{w}}(\Phi(s,a))$
  - NN is any (deep) learning network.
  - Often used in practice.

- Other parametrization (or even non parametric coding) could be used (at least in theory...).

Approximated Value Functions Usage

$$v_\pi(s) \simeq V_{\mathbf{w}_\pi}(s)$$
$$q_\pi(s,a) \simeq Q_{\mathbf{w}_\pi}(s,a)$$
$$\operatorname*{argmax}_a q_\pi(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\pi}(s,a)$$

$$v_\star(s) \simeq V_{\mathbf{w}_\star}(s)$$
$$q_\star(s,a) \simeq Q_{\mathbf{w}_\star}(s,a)$$
$$\operatorname*{argmax}_a q_\star(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\star}(s,a)$$

Approximated Value Functions Usage

- *Drop-in* replacements for all the value functions?
- Prediction and Planning?
- Quality and Stability?

Approximation Quality

$$v_\pi(s) \simeq V_{\mathbf{w}_\pi}(s)$$
$$q_\pi(s,a) \simeq Q_{\mathbf{w}_\pi}(s,a)$$
$$\operatorname*{argmax}_a q_\pi(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\pi}(s,a)$$

$$v_\star(s) \simeq V_{\mathbf{w}_\star}(s)$$
$$q_\star(s,a) \simeq Q_{\mathbf{w}_\star}(s,a)$$
$$\operatorname*{argmax}_a q_\star(s,a) \simeq \operatorname*{argmax}_a Q_{\mathbf{w}_\star}(s,a)$$

Approximation Quality Norm

- Ideal loss:

$$\|v - V_{\mathbf{w}}\|_\infty \quad \text{or} \quad \|q - Q_{\mathbf{w}}\|_\infty$$

  as this is the error used in all the previous analysis.
- Practical loss:

$$\|v - V_{\mathbf{w}}\|_{\mu,p}^p = \sum_s \mu(s)|v(s) - V_{\mathbf{w}}(s)|^p$$
$$\text{or} \quad \|q - Q_{\mathbf{w}}\|_{\mu,p}^p = \sum_{s,a} \mu(s,a)|q(s,a) - Q_{\mathbf{w}}(s,a)|^p$$

  often with $p = 2$ and μ related to the behavior policy.

Approximation Target(s)

$$q(s,a) = \mathcal{T} q(s,a) \sim Q_{\mathbf{w}}(s,a) \longrightarrow \begin{cases} \|q - Q_{\mathbf{w}}\|_{\mu,p} \text{ small} \\ \|\mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_{\mu,p} \text{ small} \end{cases}$$

Approximation Targets(s)

- Direct measurement.
- Bellman residual error.

Quality Measure

- Norm: $\|q - Q_{\mathbf{w}}\|_{\mu,p}$ or $\|\mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_{\mu,p}$ small.
- Projection (with linear parametrization): $\|P_\Phi \left(q - Q_{\mathbf{w}}\right)\|_{\mu,p}$ or $\|P_\Phi \left(\mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}\right)\|_{\mu,p}$ small
- Probes Z: $\mathbb{E}_Z[|\langle q - Q_{\mathbf{w}}, Z\rangle|^p]$ or $\mathbb{E}_Z[|\langle \mathcal{T} Q_{\mathbf{w}} - Q_{\mathbf{w}}, Z\rangle|^p]$ small.

- Lots of freedom but hard to link with optimality of derived policy!

Outline

Prediction, Approximation and Gradient Descent

$$\min_{\mathbf{w}} \sum_{s,a} \mu_b(s,a) \left| q_\pi(s,a) - Q_{\mathbf{w}}(s,a) \right|^2$$

Prediction, Approximation and Gradient Descent

- Prediction objective:
$$\overline{\mathrm{VE}}(\mathbf{w}) = \sum_{q} \mu_b(s,a) \left| q_\pi(s,a) - Q_{\mathbf{w}}(s,a) \right|^2$$
- Gradient:
$$\nabla \overline{\mathrm{VE}}(\mathbf{w}) = -2 \sum_{s,a} \mu_b(s,a) \left(q_\pi(s,a) - Q_{\mathbf{w}}(s,a) \right) \nabla Q_{\mathbf{w}}(s,a)$$
- Stochastic gradient:
$$\widehat{\nabla} \overline{\mathrm{VE}}(\mathbf{w}) = -2 \left(q_\pi(S_t, A_t) - Q_{\mathbf{w}}(S_t, A_t) \right) \nabla Q_{\mathbf{w}}(S_t, A_t)$$

- Not a practical algorithm as q_π is unknown.

Prediction, Approximation and MC

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left(G_t - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

Monte Carlo Approach

- Use $b = \pi$ and replace $q_\pi(S_t, A_t)$ by its Monte Carlo estimate G_t.
- Still a Stochastic Gradient of the original problem with limit (if it exists) satisfying
$$\begin{aligned} &\mathbb{E}_\pi[(G_t - Q_{\boldsymbol{w}_\infty}(S_t, A_t)) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)] \\ &= \mathbb{E}[(q_\pi(S_t, A_t) - Q_{\boldsymbol{w}_\infty}(S_t, A_t)) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)] = 0 \end{aligned}$$
- Convergence ensured for the linear parametrization as it is a convex problem.

- Correspond exactly to the tabular MC prediction algorithm for the tabular parametrization.
- For the linear parametrization:
  Limiting equation: $\mathbb{E}_\pi[q_\pi(S_t, A_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right] \boldsymbol{w}_\infty$

Prediction, Approximation and TD

$$\mathbf{w}_{t+1} = \mathbf{w}_t + 2\alpha_t \left(R_{t+1} + \gamma Q_{\mathbf{w}_t}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

Temporal Differencies Approach

- Use $b = \pi$ and replace $q_\pi(S_t, A_t)$ by $R_{t+1} + \gamma Q_{\mathbf{w}_t}(S_{t+1}, A_{t+1})$.
- Not a Stochastic Gradient of the original problem but a Stochastic Approximation algorithm with limit (if it exists) satisfying

$$\begin{aligned} &\mathbb{E}_\pi[(R_t + \gamma Q_{\mathbf{w}_\infty}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_\infty}(S_t, A_t)) \nabla Q_{\mathbf{w}_\infty}(S_t, A_t)] \\ &= \mathbb{E}_\pi[((\mathcal{T}^\pi Q_{\mathbf{w}_\infty} - Q_{\mathbf{w}_\infty})(S_t, A_t)) \nabla Q_{\mathbf{w}_\infty}(S_t, A_t)] = 0 \end{aligned}$$

- No simple argument to justify the convergence...

- In general, no straightforward relation with Bellman operator.
- Correspond exactly to the tabular TD prediction algorithm for the tabular parametrization.

$$\mathbf{w}_{t+1} = \mathbf{w}_t + 2\alpha_t \left(\tilde{G}_t - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

Temporal Differencies Approach

- Replace $q_\pi(S_t, A_t)$ by any advanced return $\tilde{G}_t$.
- Not a Stochastic Gradient of the original problem but a Stochastic Approximation algorithm with limit (if it exists) satisfying

$$\mathbb{E}_\pi\left[\left(\tilde{G}_t - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_\infty}(S_t, A_t)\right]$$
$$= \mathbb{E}_\pi\left[\left((\widetilde{\mathcal{T}}^\pi Q_{\mathbf{w}_\infty} - Q_{\mathbf{w}_\infty})(S_t, A_t)\right) \nabla Q_{\mathbf{w}_\infty}(S_t, A_t)\right] = 0$$

- No simple argument to justify the convergence...

- In general, no straightforward relation with Bellman operator.
- Correspond exactly to the tabular TD prediction algorithm for the tabular parametrization.

Prediction, Approximation and Eligibility Trace

$$
\begin{aligned}
z_t &= \gamma\lambda z_{t-1} + \nabla Q_{\mathbf{w}_t}(S_t, A_t) \\
\delta_t &= R_{t+1} + \gamma Q_{\mathbf{w}_t}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_t}(S_t, A_t) \\
\mathbf{w}_{t+1} &= \mathbf{w}_t + \alpha_t \delta_t z_t
\end{aligned}
$$

Eligibility Trace

- Use $b = \pi$ and rewrite the TD(λ) updates using the backward point of view.
- No strict equivalence due to time evolution of the parameterization.
- Stochastic Approximation with limit (if it exists) satisfying

$$
\begin{aligned}
&\mathbb{E}_\pi[(R_{t+1} + \gamma Q_{\mathbf{w}_\infty}(S_{t+1}, A_{t+1}) - Q_{\mathbf{w}_\infty}(S_t, A_t))\, z_t] \\
&= \mathbb{E}_\pi[(\mathcal{T}^\pi Q_{\mathbf{w}_\infty} - Q_{\mathbf{w}_\infty})\,(S_t, A_t) z_t] = 0
\end{aligned}
$$

- No simple argument to justify the convergence.

Outline

Linear Parametrization

$$Q_{\boldsymbol{w}}(S_t, A_t) = \Phi(S_t, A_t)^\top \boldsymbol{w} \quad \text{and} \quad \nabla Q_{\boldsymbol{w}}(S_t, A_t) = \Phi(S_t, A_t)$$

Linear Parametrization

- Extension of the tabular setting.
- Derivative is independent of $\boldsymbol{w}$.
- Analysis of Stochastic Approximation often possible!
- More than a toy model as an algorithm not converging in the linear case will almost certainly not converge in a more general setting.

Linear Parametrization and MC

$$\text{Iteration:}\,\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t \left(G_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t\right) \Phi(S_t, A_t)$$

$$\text{Limiting equation: } \mathbb{E}_\pi[q_\pi(S_t, A_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right] \boldsymbol{w}_\infty$$

$$\text{ODE: } \frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right] (\boldsymbol{w} - \boldsymbol{w}_\infty)$$

Linear Parametrization and MC

- Limiting equation is a linear equation.
- Under asymptotic stationarity assumption, convergence of ODE as $\mathbb{E}_\pi\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right]$ is a Gram Matrix with positive eigenvalues (provided Φ is not redundant and under an ergodicity assumption).
- Need to explore all state-action pairs!

Episodic

Linear Parametrization and TD

Iteration: $\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t \left(R_{t+1} + \gamma \Phi(S_{t+1}, A_{t+1})^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t\right) \Phi(S_t, A_t)$

Lim. eq.: $\mathbb{E}_\pi[r(S_T, A_t)\Phi(S_t, A_t)] = \mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma\Phi(S_{t+1}, A_{t+1})^\top\right)\right] \boldsymbol{w}_\infty$

ODE: $\dfrac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma\Phi(S_{t+1}, A_{t+1})^\top\right)\right](\boldsymbol{w} - \boldsymbol{w}_\infty)$

Linear Parametrization and TD

- Convergence of ODE if $\mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma\Phi(S_{t+1}, A_{t+1})^\top\right)\right]$ has complex eigenvalues with positive real parts...
- which can be proved to be true under an ergodicity assumption!
- Need to explore all state-action pairs!
- Different solution than MC! Minimization of the Projected Bellman Residual...
- **Prop:**

$$\overline{VE}(\boldsymbol{w}_{\mathrm{TD}}) \leq \frac{1}{1-\gamma}\overline{VE}(\boldsymbol{w}_{\mathrm{MC}}) = \frac{1}{1-\gamma}\min_{\boldsymbol{w}} \overline{VE}(\boldsymbol{w})$$

Least-Squares TD

$$b = \mathbb{E}_\pi[r(S_T, A_t)\Phi(S_t, A_t)] \sim \frac{1}{t}\sum_{t'=0}^{t-1} R_{t'+1}\Phi(S_{t'}, A_{t'})$$

$$\begin{aligned} A &= \mathbb{E}_\pi\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma\Phi(S_{t+1}, A_{t+1})^\top\right)\right] \\ &\sim \frac{1}{t}\sum_{t'=0}^{t-1} \Phi(S_{t'}, A_{t'})\left(\Phi(S_{t'}, A_{t'})^\top - \gamma\Phi(S_{t'+1}, A_{t'+1})^\top\right)\end{aligned}$$

Least-Squares TD

- Bypass the Stochastic Approximation scheme by estimating directly its limit:
$$\boldsymbol{w}_\infty = A^{-1}b$$
- Much more sample efficient.
- Recursive implementation possible.
- Recursive implementation maintaining an estimate of A^{-1} is also possible.

Advanced Returns

$$\text{Return: } \tilde{G}_t = \tilde{R}_{t+1} + \tilde{\Phi}_t^\top \boldsymbol{w} \quad \text{(affine formula)}$$

$$\text{Iteration: } \boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t \left(\tilde{R}_t + \tilde{\Phi}_t^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t \right) \Phi(S_t, A_t)$$

$$\text{Lim. eq.: } \mathbb{E}_\pi \left[\tilde{R}_t \Phi(S_t, A_t) \right] = \mathbb{E}_\pi \left[\Phi(S_t, A_t) \left(\Phi(S_t, A_t)^\top - \Phi_t{}^\top \right) \right] \boldsymbol{w}_\infty$$

$$\text{ODE: } \frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_\pi \left[\Phi(S_t, A_t) \left(\Phi(S_t, A_t)^\top - \tilde{\Phi}_t^\top \right) \right] (\boldsymbol{w} - \boldsymbol{w}_\infty)$$

Linear Parametrization and TD

- Convergence of ODE if $\mathbb{E}_\pi \left[\Phi(S_t, A_t) \left(\Phi(S_t, A_t)^\top - \tilde{\Phi}_t^\top \right) \right]$ has complex eigenvalues with positive real parts...
- which can be proved to be true for the advanced returns under an ergodicity assumption!

Outline

On-Policy Prediction

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + 2\alpha_t \left(\tilde{G}_t - Q_{\boldsymbol{w}_t}(S_t, A_t)\right) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

On-line TD Algorithm

- Use the policy Π to obtain the interactions $S_t A_t R_{t+1} S_{t+1} A_{t+1} \dots$
- Convergence... for linear parametrization under stationarity and coverage assumptions!
- Appear to *converge* even with more complex parametrization.

- Monte Carlo can be used for short episodes.
- Similar observations for elegibility trace.

On-Policy Control

$$\mathbf{w}_{t+1} = \mathbf{w}_t + 2\alpha_t \left(\tilde{G}_t - Q_{\mathbf{w}_t}(S_t, A_t) \right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$
$$\pi_{t+1}(s) = \operatorname{argmax} Q_{\mathbf{w}_t}(s, \cdot) \quad \text{(plus exploration)}$$

On-Policy Control

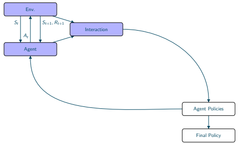
- SARSA type algorithm: update Q values and policy π while using policy π.
- Not a Stochastic Approximation algorithm anymore...
- Not approximate policy improvement as no sup-norm control...
- No proof of convergence... but appear to work well in practice.

- Non trivial scheduling issue in the definition of $\tilde{G}_t$.
- More constraints with eligibility trace.

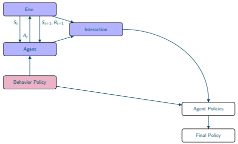
Outline

On-Policy vs Off-Policy

From



to



On-Policy vs Off-Policy

- On-Policy: the policy b used to interact is the same than the policy Π evaluated or optimized.
- Off-Policy: the policy b used to interact may be different from the policy Π evaluated or optimized.

- Off-Policy correction available for the return.

Off-Policy Prediction

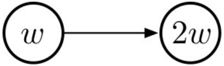
$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha_t \left(\tilde{G}_t - Q_{\mathbf{w}_t}(S_t, A_t)\right) \nabla Q_{\mathbf{w}_t}(S_t, A_t)$$

Off-policy TD Algorithm

- Use a policy b to obtain the interactions $S_t A_t R_{t+1} S_{t+1} A_{t+1} \ldots$
- Compute an (importance-sampling based) corrected return.
- Use it in the algorithm.

- **Can fail spectacularly!**
- Monte Carlo will work.

Off-Policy Divergence



Simplest Example?

- Simple transition with a reward 0.
- TD error:

$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V_{\boldsymbol{w}_t}(S_{t+1}) - V_{\boldsymbol{w}_t}(S_t)\\ &= 0 + \gamma 2\boldsymbol{w}_t - \boldsymbol{w}_t = (2\gamma - 1)\boldsymbol{w}_t\end{aligned}$$

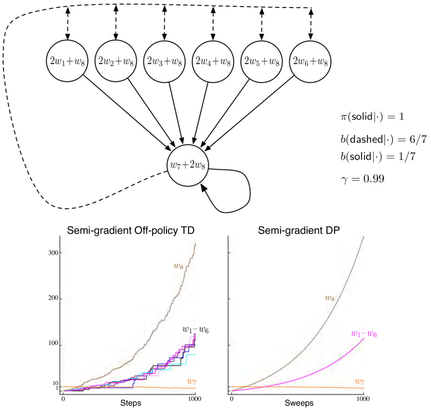
- Off-policy semi-gradient TD(0) update:

$$\begin{aligned}\boldsymbol{w}_{t+1} &= \boldsymbol{w}_t + \alpha_t \rho_t \delta_t \nabla V(S_{t+1}, \boldsymbol{w}_t)\\ &= \boldsymbol{w}_t + \alpha_t \times 1 \times (2\gamma - 1)\boldsymbol{w}_t = (1 + \alpha_t(2\gamma - 1))\boldsymbol{w}_t\end{aligned}$$

- Explosion if this transition is explored without $\boldsymbol{w}$ being update on other transitions as soon as $\gamma > 1/2$.

- No explosion if each update is followed by an update on the other state (with $\delta_t = -2\boldsymbol{w}_t$)!

Off-Policy Divergence



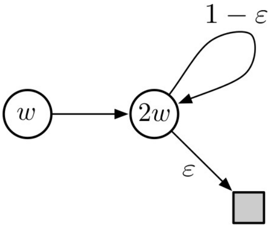
Baird's Counterexample

- Divergence of off-policy algorithm even without sampling, i.e. in Dynamic Programming.

Discounted

Source: Sutton and Barto

Off-Policy Divergence



Tsistiklis and Van Roy's Counterexample

- Exact minimization of bootstrapped $\overline{VE}$ at each step:

$$\begin{aligned} \boldsymbol{w}_{t+1} &= \operatorname*{argmin}_{\boldsymbol{w}} \sum_{s} \left(V_{\boldsymbol{w}_t}(s) - \mathbb{E}_\pi[R_{t+1} + \gamma V_{\boldsymbol{w}_t}(S_{t+1}) | S_t = s]\right)^2 \\ &= \operatorname*{argmin}_{\boldsymbol{w}} (\boldsymbol{w} - \gamma 2\boldsymbol{w}_t)^2 + (2\boldsymbol{w} - (1-\epsilon)\gamma 2\boldsymbol{w}_t)^2 \\ &= \frac{6-4\epsilon}{5}\gamma \boldsymbol{w}_t \end{aligned}$$

- Divergence if $\gamma > 5/(6-4\epsilon)$.

Linear Parametrization and TD

Iteration: $\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha_t(R_{t+1} + \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top \boldsymbol{w}_t - \Phi(S_t, A_t)^\top \boldsymbol{w}_t)\Phi(S_t, A_t)$

Lim. Eq.: $\mathbb{E}_b[r(S_T, A_t)\Phi(S_t, A_t)] = \mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top\right)\right]\boldsymbol{w}_\infty$

ODE: $\frac{d\boldsymbol{w}}{dt} = -\mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top\right)\right](\boldsymbol{w} - \boldsymbol{w}_\infty)$

Linear Parametrization and TD

- Convergence of ODE if
$$\mathbb{E}_b\left[\Phi(S_t, A_t)\left(\Phi(S_t, A_t)^\top - \gamma \sum_a \pi(a|S_{t+1})\Phi(S_{t+1}, a)^\top\right)\right] = \Phi\Xi(I - \gamma P^\pi)\Phi^\top$$
(with $\Phi = (\Phi(s, a))$, $\Xi = \mathrm{diag}(\mu_b(s, a))$) and $P\pi$ the transition matrix associated to π) has complex eigenvalues with positive real parts...
- Proof for on-policy relies on $\mu_b = \mu_\pi$ which satisfies $\mu_\pi{}^\top P_\pi = \mu_\pi{}^\top$.
- Not true anymore with an arbitrary behavior policy!

Discounted

Deadly Triad

Deadly Triad

- **Function approximation**
- **Bootstrapping**
- **Off-policy training**

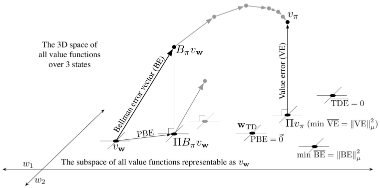
- **Instabilities as soon as the three are present!**

Issue

- Function approximation is unavoidable.
- Bootstrap is much more computational and data efficient.
- Off-policy may be avoided... but essential when dealing with extended setting (learn from others or learn several tasks)

- Dead End?

Target?



Linear Parametrization Target?

- Prediction objective $\overline{VE}$:

$$\|q_\pi - Q_{\mathbf{w}}\|_\mu^2$$

- Bellman Error $\overline{BE}$:

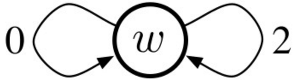
$$\|\mathcal{T}^\pi Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_\mu^2$$

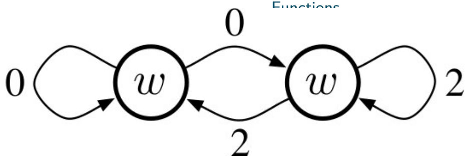
- Projected Bellman Error $\overline{PBE}$:

$$\|\operatorname{Proj} \mathcal{T}^\pi Q_{\mathbf{w}} - Q_{\mathbf{w}}\|_\mu^2$$

with $\operatorname{Proj} = \Phi(\Phi^\top \Xi \Phi)\Phi\,(\Phi)\,\Xi$.

Prediction Objective





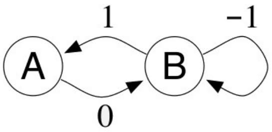
Prediction Objective

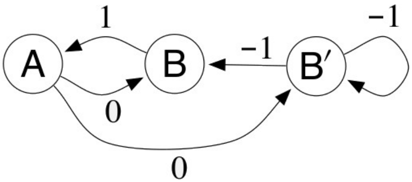
- Two MRP with the same outputs (because of approximation).
- but different $\overline{VE}$.
- Impossibility to learn $\overline{VE}$.
- Minimizer however is learnable:

$$\begin{aligned}\overline{RE}(\boldsymbol{w}) &= \mathbb{E}\left[(G_t - V_{\boldsymbol{w}_t}(S_t))^2\right] \\ &= \overline{VE}(\boldsymbol{w}) + \mathbb{E}\left[(G_t - v_\pi(S_t))^2\right]\end{aligned}$$

- MC method target.

Bellman Error



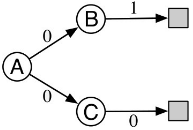


Bellman Error

- Two MRP with the same outputs (because of approximation).
- Different $\overline{BE}$.
- Different minimizer!
- $\overline{BE}$ is not learnable!

TD Error

$$\overline{TDE}(\boldsymbol{w}) = \|\mathbb{E}_\pi\left[\delta_t^2 | S_t, A_t\right]\|_\mu$$



Mean-Squares TD Error

- $\overline{TDE}(\boldsymbol{w}) = \mathbb{E}_b\left[\rho_t \delta^2\right]$
- Gradient: $\nabla \overline{TDE}(\boldsymbol{w}) =$ $\mathbb{E}_b[\rho_t\left(R_t + \gamma Q_{\boldsymbol{w}}(S_{t+1}, A_{t+1})\right) - Q_{\boldsymbol{w}_t}(S_t, A_t))\left(\gamma \nabla Q_{\boldsymbol{w}_t}(S_{t+1}, A_{t+1}) - \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)\right)]$
- SGD algorithm. . .
- but solutions often converge to not to a *desirable place* even without approximation!

Projected Bellman Error

$$\| \operatorname{Proj} \mathcal{T}^\pi Q_{\boldsymbol{w}} - Q_{\boldsymbol{w}} \|_\mu^2 \quad \text{with } \operatorname{Proj} = \Phi(\Phi^\top \Xi \Phi)^{-1} \Phi^\top \Xi.$$

Projected Bellman Error

- Rewriting

$$\begin{aligned} \overline{PBE}(\boldsymbol{w}) &= \| \operatorname{Proj} \mathcal{T}^\pi q_{\boldsymbol{w}} - q_{\boldsymbol{w}} \|_\mu^2 = \| \operatorname{Proj} \delta_{\boldsymbol{w}} \|_\mu^2 \\ &= (\operatorname{Proj} \delta_{\boldsymbol{w}})^\top \Xi (\operatorname{Proj} \delta_{\boldsymbol{w}}) = (\Phi^\top \Xi \delta_{\boldsymbol{w}})^\top \left(\Phi^\top \Xi \Phi \right)^{-1} (\Phi^\top \Xi \delta_{\boldsymbol{w}}) \end{aligned}$$

- Gradient:

$$\nabla \overline{PBE}(\boldsymbol{w}) = 2 \nabla (\Phi^\top \Xi \delta_{\boldsymbol{w}})^\top \left(\Phi^\top \Xi \Phi \right)^{-1} (\Phi^\top \Xi \delta_{\boldsymbol{w}})$$

- Expectations:

$$\begin{aligned} \Phi^\top \Xi \delta_{\boldsymbol{w}} &= \mathbb{E}_b[\rho_t \delta_t \Phi(S_t, A_t)] \\ \nabla (\Phi^\top \Xi \delta_{\boldsymbol{w}})^\top &= \mathbb{E}_b\left[\rho_t (\gamma \Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t)) \Phi(S_t, A_t)^\top\right] \\ \Phi^\top \Xi \Phi &= \mathbb{E}_b\left[\Phi(S_t, A_t) \Phi(S_t, A_t)^\top\right] \end{aligned}$$

- Not yet a SGD/SA as the gradient is a product of several terms...

Projected Bellman Error

Gradient and Stochastic Approximation

- Gradient:
$$\nabla \overline{PBE}(\boldsymbol{w}) = 2\mathbb{E}_b\left[\rho_t(\gamma\Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t))\Phi(S_t, A_t)^\top\right]$$
$$\left(\mathbb{E}_b\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right]\right)^{-1} \mathbb{E}_b[\rho_t\delta_t\Phi(S_t, A_t)]$$
- Least-squares inside:
$$v = \left(\mathbb{E}_b\left[\Phi(S_t, A_t)\Phi(S_t, A_t)^\top\right]\right)^{-1} \mathbb{E}_b\left[\rho_t\delta_t\Phi(S_t, A_t)^\top\right]$$
$$\Leftrightarrow v = \operatorname*{argmin}_v \mathbb{E}_b\left[\left(\Phi(S_t, A_t)^\top v_t - \rho_t\delta_t\right)^2\right]$$

  which can be estimated by
$$v_{t+1} = v_t + \beta_t\Phi(S_t, A_t)(\delta_t - \rho_t\Phi(S_t, A_t)^\top v_t)$$
- Plugin pseudo gradient (SA):
$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t\rho_t(\gamma\Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t))\Phi(S_t, A_t)^\top v_t$$
- Same target than Pseudo Gradient but converging algorithm provided $\alpha_t \ll \beta_t$.

Discounted

Gradient TD Algorithm

GTD

- Simultaneous update:

$$v_{t+1} = v_t + \beta_t \Phi(S_t, A_t)(\delta_t - \rho_t \Phi(S_t, A_t)^\top v_t)$$

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t \rho_t (\gamma \Phi(S_{t+1}, A_{t+1}) - \Phi(S_t, A_t))\Phi(S_t, A_t)^\top v_t$$

- As $\alpha_t \ll \beta_t$, $\boldsymbol{w}$ is seen as constant by v...

TDC

- Simultaneous update:

$$v_{t+1} = v_t + \beta_t \Phi(S_t, A_t)(\delta_t - \rho_t \Phi(S_t, A_t)^\top v_t)$$

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - 2\alpha_t \rho_t (\delta_t \Phi(S_t, A_t) - \gamma \Phi(S_{t+1}, A_{t+1}))\Phi(S_t, A_t)^\top v_t$$

- Obtained by a similar derivation but faster in practice...
- As $\alpha_t \ll \beta_t$, $\boldsymbol{w}$ is seen as constant by v...

- Restricted to the linear setting but interesting insight.

Outline

Stochastic Approximation

$$\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\implies \theta_k \to \{\theta, H(\theta) = 0\}$$

Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\text{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - the algorithm converges if we replace h_k by H.

- Convergence toward a neighborhood if α is kept constant (as often in practice).
- Most famous example are probably Robbins-Monro and SGD.
- Proof quite technical in general.
- The convergence with H is easy to obtain for a contraction.

Stochastic Approximation and ODE

From $\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k)$ with $h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$

to $\dfrac{d\tilde{\theta}}{dt} = H(\tilde{\theta})$

ODE Approach

- General proof showing that the algorithm converges provided the ODE converges.
- Rely on the rewriting the equation
$$\frac{\theta_{k+1} - \theta_k}{\alpha_k} = h_k(\theta_k) = H(\theta_k) + \epsilon_k + \eta_k$$
- α_k can be interpreted as a time difference allowing to define a time $t_k = \sum_{t' \leq t} \alpha_k$.
- $\theta(t)$ is piecewise affine and defined through its derivative at time $t \in (t_k, t_{k+1})$.
- This piecewise function remains close to any solution of the ODE starting from θ_k for an arbitrary amount of time provided k is large enough.

- More general proofs based on martingale.

Stochastic Approximation

$$\begin{cases}\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k, \nu_k) \\ \nu_{k+1} = \nu_k + \beta_k g_k(\theta_k, \nu_k)\end{cases} \quad \text{with} \quad \begin{cases} h_k(\theta, \nu) = H(\theta, \nu) + \epsilon_k + \eta_k \\ g_k(\theta, \nu) = G(\theta, \nu) + \epsilon'_k + \eta'_k\end{cases}$$

$$\implies \theta_k \to \{\theta, H(\theta, \nu(\theta)) = 0, \nu(\theta) \in \{\nu, G(\theta, \nu) = 0\}\}$$

Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\text{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - $\sum_k \beta_k \to \infty$ and $\sum_k \beta_k^2 < \infty$,
  - $\alpha_k / \beta_k \to 0$ (two-scales assumption),
  - the algorithm converges if we replace h_k and g_k by H and G.

- Convergence toward a neighborhood if $\alpha \ll \beta$ are kept constant (as often in practice).

Stochastic Approximation and ODE

From $\begin{cases} \theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k, \nu_k) \\ \nu_{k+1} = \nu_k + \beta_k + g_k(\theta_k, \nu_k) \end{cases}$ with $\begin{cases} h_k(\theta, \nu) = H(\theta, \nu) + \epsilon_k + \eta_k \\ g_k(\theta, \nu) = G(\theta, \nu) + \epsilon'_k + \eta'_k \end{cases}$

to $\dfrac{d\tilde{\theta}}{dt} = H(\tilde{\theta}, \tilde{\nu}(\tilde{\theta}))$ with $\tilde{\nu}(\theta)$ the limit of $\dfrac{d\tilde{\nu}}{dt} = G(\theta, \tilde{\nu})$

ODE Approach

- General proof showing that the algorithm converges provided the two ODE converge.
- Quite generic setting and source of new algorithm or insight on existing ones.
- Importance of having two scales…

- Can be used to prove the convergence of GTD and TDC!

Outline

Simplified Deep Q-Learning

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \beta_t \big(R_{t+1} + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\boldsymbol{w}_t}(S_t, A_t)\big) \nabla Q_{\boldsymbol{w}_t}(S_t, A_t)$$

$$\nu_t = \boldsymbol{w}_{\lceil t/T \rceil T}$$

Simplified Deep Q-Learning

- Stochastic Approximation for a fixed ν:
  - Limiting equation:
    $$\mathbb{E}_b[(\mathcal{T}^\star Q_\nu(S_t, A_t) - Q_{\boldsymbol{w}_\infty}(S_t, A_t)) \nabla Q_{\boldsymbol{w}_\infty}(S_t, A_t)] = 0$$
  - Stochastic Gradient Descent of
    $$\mathbb{E}_b\left[(\mathcal{T}^\star Q_\nu(S_t, A_t) - Q_{\boldsymbol{w}}(S_t, A_t))^2\right]$$
  - $Q_{\boldsymbol{w}} \to \mathcal{T}^\star Q_\nu$
- Approximate Value Iteration Scheme!

- Two-scales algorithm flavour as ν is kept constant.
- Explicit two scales with $\nu_{t+1} = \nu_t + \alpha_t(\boldsymbol{w}_t - \nu_t)$ variation.
- Could be used for prediction with $R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q_{\nu_t}(S_{t+1}, a)$

Deep Q-Learning

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \beta_t (R_t + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\mathbf{w}}(S_t, A_t)) \nabla Q_{\mathbf{w}}(S_t, A_t)$$

$$\nu_t = \mathbf{w}_{\lceil t/T \rceil T}$$

- **Who are $S_t, A_t, R_{t+1}, S_{t+1}$?** and thus to what corresponds $\mathbb{E}_b$?

Simplified Deep Q-Learning

- Use a behaviour policy b.
- The current greedy plus exploration Q-policy can be used.

Neural Fitted-Q

- Instead of a policy b, use a fix dataset $\mathcal{D}$ of $S_t, A_t, R_{t+1}, S_{t+1}$.
- Several pass on the data can be made.

Deep Q-Learning

- Use the current greedy plus exploration Q-policy to populate a FIFO buffer $\mathcal{D}$.
- Use random samples of the buffer $\mathcal{D}_t$ (more than one per interaction is OK).

Deep Q-Learning

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \beta_t (R_t + \gamma \max_a Q_{\nu_t}(S_{t+1}, a) - Q_{\mathbf{w}}(S_t, A_t)) \nabla Q_{\mathbf{w}}(S_t, A_t)$$

$$\nu_t = \mathbf{w}_{\lceil t/T \rceil T}$$

Plus tricks

Deep Q-Learning Tricks

- Replay buffer
- Double Q-Learning
- Better Exploration
- Advanced Return and Distributional
- Network Architecture

- Rainbow paper. . .

Replay Buffer

Replay Buffer

- Replace an expectation over real trajectories by an empirical average over past (short) sub-trajectories stored in a replay buffer.
- The empirical average corresponds to uniform sampling.
- If the policy is changing across time, we should use a importance sampling correction to be faithful with the theory...
- Not necessary for one-step Q learning but required for most of the other methods where replay buffer is used.
- Often no correction in practice if the policies used in the buffer are closed to the current one.
- Prioritized sweeping variant possible...

- Buffer can be constructed in parallel of the learning part.
- Only requires to transmit the *current* greedy plus exploration Q-policy.

Double Q-Learning

Q-Learning and overestimation

- Target: $R_{s,a} + \gamma \max_{a'} Q_{\mathbf{w}}(s', a')$
- Approximation issue: $Q_{\mathbf{w}}(s', a') \sim Q(s, a) + \epsilon(s, a)$
- Consequence: $\mathbb{E}[\max_a Q_{\mathbf{w}}(S_t, a)] \geq \max\left(Q(s, a) + \mathbb{E}[\epsilon(s, a)]\right)$

Double Q-Learning with two Q functions: $Q_{\mathbf{w}_1}$ and $Q_{\mathbf{w}_2}$

- Used in a crossed way for the target of $Q_{\mathbf{w}_i}$:

$$R_{s,a} + \gamma Q_{\mathbf{w}_{i'}}(s', \operatorname*{argmax}_{a'} Q_{\mathbf{w}_i}(s', a'))$$

- Mitigates the bias.

Clipped Q-Learning with several Q functions: $Q_{\mathbf{w}_i}$

- Used in a pessimistic way for the target of $Q_{\mathbf{w}_i}$:

$$R_{s,a} + \gamma \min_{i'} Q_{\mathbf{w}_{i'}}(s', \operatorname*{argmax}_{a'} Q_{\mathbf{w}_i}(s', a'))$$

- Seems even more efficient.

Outline

Continuous Action

- Case (almost) not yet covered in the lectures.
- Most complex theoretical extension.

Prediction

- No algorithmic issue if one can sample π.
- Off-policy can be considered under a domination assumption.

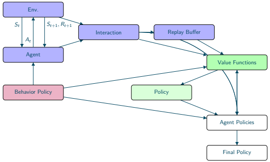
Planning

- Main issue is the argmax of the greedy policy (or the sampling of Gibbs policy).
- May be impossible to compute.
- Possible if the parametrization of Q with respect to a is simple (e.g. explicit quadratic dependency in a).
- An alternative could be to approximate the argmax operator, or to learn how to approximate the argmax directly, which is very close to approximating directly the policy itself. . .

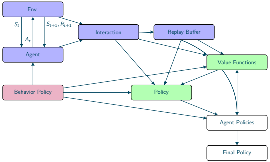
Outline

Policy Point of View

From



to



Policy Point of View

- Optimize policy directely instead of deriving it from a value function.
- Avoid the argmax operator.
- Most natural POV?

- Pontryagin vs Hamilton-Jacobi(-Bellman) in control!

Outline

Policy and Goal

$$J_\mu(\pi) = \sum_s \mu(s) v_\pi(s)$$

Goal: average expected return over the states

- Target used to define the linear programming formulation of an optimal policy in the tabular setting.
- μ can be the initial distribution of the states (independent of π)...
- but may also depends on π (for instance the associated stationary measure)
- Other choices will appear.

- Goal: optimize $J_\mu(\pi)$ in π!

Parametric Policy

$$\pi_\theta(a|s) = \begin{cases} \frac{e^{h_\theta(a,s)}}{\sum_{a'} e^{h_\theta(a,s')}} & \text{(softmax)} \\ P_{h_\theta(s)}(a) & \text{(parametric conditional model)} \\ \mathbf{1}_{a=h_\theta(s)} & \text{(deterministic)} \end{cases}$$

Parametric Policy

- Restriction of the set of policy to a parametrized one.
- Most classical parametrizations:
  - Soft-max with a preference function $h_\theta(a, s)$,
  - Parametric conditional model with parameter $h_\theta(s)$
- To be useful need to be able to sample the distribution.
- h_θ: from linear model to deep learning...
- Most of our result will assume that $\pi_\theta(a|s)$ is differentiable with respect to θ.

- Deterministic policies will be considered with a different analysis.

Episodic Setting: Gradient of Expected Returns

$$v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}[G_0 | S_0 = s]$$

$$\nabla_\theta v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0 \middle| S_0 = s\right]$$

Expected Returns

- Rely on $v_{\pi_\theta}(s) = \sum_\tau \mathbb{P}_{\pi_\theta}(\tau | S_0 = s)\, G_0(\tau)$ and

$$\begin{aligned}
\nabla \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) &= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \nabla \log \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \\
&= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t (\nabla \log \pi_\theta(A_t|S_t) + \nabla p(R_{t+1}, S_{t+1}|S_t, A_t)) \\
&= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t \nabla \log \pi_\theta(A_t|S_t)
\end{aligned}$$

- In an episodic setting, any trajectory τ ends at a finite time T_τ.

Episodic Setting: Policy Gradient Theorem

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t | S_t)\right) G_0\right]$$

Policy Gradient Theorem

- Natural μ: initial state distribution.
- Gradient is an expectation: MC type algorithm. . .

- Can be interpreted as the gradient of a the maximum likelihood of the actions weighted by the return.
- Favors good actions over bad ones.

Baseline and Variance Reduction

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

Variance Reduction and Baseline

- The previous formulae are valid if one replace G_0 by any function of τ.
- For any constant b, this leads to

$$\nabla \mathbb{E}_{\pi_\theta}[b] = 0 = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) b\right]$$

- Optimal value for
$b = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)^2 G_0\right] / \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)^2\right]$
- Most used value $b = \mathbb{E}_{\pi_\theta}[G_0]$.

Gradient(s) of Expected Return

$$v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}\left[\sum \gamma^t R_t \middle| S_0 = s\right]$$

$$\nabla v_{\pi_\theta}(s) = \sum_t \gamma^t \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t'=0}^{t-1} \nabla \log \pi_\theta(A_{t'}|S_{t'})\right) R_t \middle| S_0 = s\right]$$

$$= \sum_{t'} \mathbb{E}_{\pi_\theta}\left[\nabla \log \pi_\theta(A_{t'}|S_{t'}) \left(\sum_{t \geq t'} \gamma^t R_t\right) |S_0 = s\right]$$

$$= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s]$$

$$= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}\left[\nabla \log \pi_\theta(A_{t'}|S_{t'}) \underbrace{(q_{\pi_\theta}(S_{t'}, A_{t'}) - v_{\pi_\theta}(S_{t'}))}_{a_{\pi_\theta}(S_{t'}, A_{t'})} |S_0 = s\right]$$

From Returns to Value Functions

- Action point of view and use of value functions.

More Gradient(s)

$$\nabla v_{\pi_\theta}(s) = \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s]$$

$$= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) a_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s]$$

$$= \sum_{s'} \left(\sum_{t} \gamma^{t} \mathbb{P}_{\pi_\theta}(S_t = s'|S_0 = s) \right) \left(\sum_{a} \pi_\theta(a|s') \nabla \log \pi_\theta(a|s') q_{\pi_\theta}(s', a) \right)$$

$$= \sum_{s'} \left(\sum_{t} \gamma^{t} \mathbb{P}_{\pi_\theta}(S_t = s'|S_0 = s) \right) \left(\sum_{a} \pi_\theta(a|s') \nabla \log \pi_\theta(a|s') a_{\pi_\theta}(s', a) \right)$$

Focus on states

- Even more stochastic gradients!

Policy Gradient(s)

$$J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right)$$

$$= \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s, a) - v_{\pi_\theta}(s, a)) \right)$$

Discounted Setting

- Average (discounted) return from the beginning.
- Focus on early steps in discounted setting...

Policy Improvement Lemma

$$J_{\mu_0}(\pi') - J_{\mu_0}(\pi) = \sum_s \sum_t \gamma^t \mathbb{P}_{\pi'}(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, q_\pi(s, a) \right)$$

$$= \sum_s \sum_t \gamma^t \mathbb{P}_{\pi'}(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s, a) \right)$$

Proof

- By construction, if S_t is a trajectory using policy π':

$$v_{\pi'}(S_t) - v_\pi(S_t) = \sum_a (\pi'(a|S_t) - \pi(a|S_t))\, q_\pi(S_t, a) + \sum_a \pi'(a|s_t)\, (q_{\pi'}(S_t, a) - q_\pi(S_t, a))$$

$$= \sum_a (\pi'(a|s_t) - \pi(a|S_t))\, v_\pi(S_t, a) + \mathbb{E}_{\pi'}[v_{\pi'}(S_{t+1}) - v_\pi(S_{t+1})|S_t]$$

- Discounted setting shortcut

$$v_{\pi'} - v_\pi = r_{\pi'} + \gamma P^{\pi'} v_{\pi'} - r_\pi - \gamma P^\pi v_\pi = r_{\pi'} - r_\pi + \gamma \left(P^{\pi'} - P^\pi \right) v_\pi + \gamma P^{\pi'} (v_{\pi'} - v_\pi)$$

$$v_{\pi'} - v_\pi = (I - \gamma P^{\pi'})^{-1} \left(r_{\pi'} - r_\pi + \gamma \left(P^{\pi'} - P^\pi \right) v_\pi \right)$$

Approximate Policy Improvement Lemma

$$\left| J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_s \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right) \right|$$

$$= \left| \sum_s \sum_t \gamma^t \left(\mathbb{P}_{\pi'}(S_t = s) - \mathbb{P}_\pi(S_t = s)\right) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right) \right|$$

$$\leq \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Approximate Policy Improvement Lemma

- If $\max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \leq \epsilon$

$$\mathbb{P}_{\pi'}(S_t = s) = (1-\epsilon)^t \mathbb{P}_\pi(S_t = s) + (1 - (1-\epsilon)^t)\mathbb{P}_{\text{mistake}}(S_t = s)$$

$$\rightarrow |\mathbb{P}_{\pi'}(S_t = s) - \mathbb{P}_\pi(S_t = s)| \leq 2(1 - (1-\epsilon)^t) \leq 2\epsilon t$$

- $\sum_t 2\gamma^t t = \frac{2\gamma}{(1-\gamma)^2}$

Approximate Policy Improvement Lemma

$$\left| J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_s \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right) \right|$$

$$\leq \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Approximate Policy Improvement Lemma and Policy Gradient Theorem

- Let $\pi' = \pi_{\theta+h}$ and π_θ
  - $\pi_{\theta+h}(a|s) - \pi_\theta(a|s) = \pi_\theta(a|s)\langle \nabla \log \pi_\theta(a|s), h\rangle + O(\|h\|^2)$
  - $\|\pi_{\theta+h}(\cdot|s) - \pi_\theta(\cdot|s)\|_1 \leq \|h\| \max_a \|\nabla \log \pi_\theta(a|s)\| + O(\|h\|^2)$
- Implies Policy Gradient Theorem:

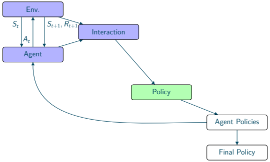
$$J_{\mu_0}(\pi_{\theta+h})$$

$$= J_{\mu_0}(\pi_\theta) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \left(\sum_a \pi_\theta(a|s) \langle \nabla \log \pi_\theta(s,a), h\rangle a_\pi(s,a) \right) + O(\|h\|^2)$$

Discounted

Outline

Monte Carlo Approach



$$G_t = \sum_{t' \geq t} R_{t+1}$$

$$Q_{t,\pi_\theta}(s,a) = \mathbb{E}[G_t | S_t = s, A_t = a]$$

Monte Carlo

- Replace every return by an empirical estimate along episodes.
- Need to wait until the end of the episods.

Episodic

REINFORCE: Monte Carlo Based Policy Gradient

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0\right]$$

$$= \sum_s \left(\sum_t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s,a)\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0 \quad \text{or} \quad \widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) G_t$$

REINFORCE

- Plain MC (SGD) algorithm.
- Need to wait until the end of the episods.
- Convergence guarantees (even in off-line setting with importance sampling).

REINFORCE with Baseline

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

$$= \sum_s \left(\sum_t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)\,(q_{\pi_\theta}(s,a) - b(s))\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)$$

or $$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t)\,(G_t - b(S_t))$$

REINFORCE with baseline

- Several choices for b...
- and for $b(s)$ which can be any function (a crude estimate of $V_{t,\pi}(s)$ for instance)!
- Convergence guarantees (even in off-line setting with importance sampling).

Discounted REINFORCE?

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

$$= \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)\left(q_{\pi_\theta}(s,a) - b(s)\right)\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)$$

or $$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t)\left(G_t - b(S_t)\right)$$

Discounted REINFORCE

- Can be defined...
- but still requires an episodic setting for the discounted return G_t to be computed.

Discounted Measure?

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t) \left(G_t - b(S_t)\right)$$

$$\longrightarrow \widehat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) = \frac{1}{1-\gamma} \nabla \log \pi_\theta(A_t|S_t) \left(G_t - b(S_t)\right)?$$

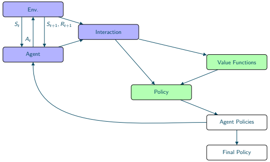
Discounted Measure?

- Much less weights for later states if μ corresponds to the initial state distribution!
- Equal weights corresponds to an averaged probability independent t, which is well defined if the initial distribution is the stationary distribution μ_{π_θ} corresponding to π_θ (it it exists).
- Approximately true after a burning stage if we reach stationarity. . .
- Better handled by the average return!

- More on this later. . .

Outline

Actor/Critic



Actor/Critic

- Actor: Parametric policy π_θ used.
- Critic: Q-value function $Q_{\mathbf{w}}(\cdot, \cdot)$ approximating Q_{π_θ}.
- Critic follows the Actor, which is optimized using the Critic.

- In Value Approximation, the Actor follows the Critic (through the argmax operator).
- In on-line methods, the Actor is used to interact with the environment.

Actor/Critic

$$J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s,a) - v_{\pi_\theta}(s,a)) \right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(q_{\pi_\theta}(S_t, A_t) - \sum_a \pi(a|S_t) q_{\pi_\theta}(S_t, A_t) \right)$$

$$\simeq \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(Q_{\mathbf{w}}(S_t, A_t) - \sum_a \pi(a|S_t) Q_{\mathbf{w}}(S_t, A_t) \right)$$

Actor/Critic

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating q_{π_θ}.
- Requires a two-scales algorithm so that $Q_{\mathbf{w}}$ is always a good estimate of q_{π_θ}.

- Is this a real algorithm in a non-episodic setting?

Actor/Critic

$$J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \mu_{\pi_\theta}(s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \frac{1}{1-\gamma} \mathbb{P}_{\pi_\theta}(S_t = s) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s,a) - v_{\pi_\theta}(s,a)) \right)$$

$$\widehat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) \simeq \frac{1}{1-\gamma} \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(Q_{\mathbf{w}}(S_t, A_t) - \sum_a \pi(a|S_t) Q_{\mathbf{w}}(S_t, A_t) \right)$$

Actor/Critic

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating q_{π_θ}.
- Requires a two-scales algorithm so that $Q_{\mathbf{w}}$ is always a good estimate of q_{π_θ}.

- Require the existence of a stationary measure... and that this stationary measure is reached *quickly*.
- Much harder to do off-policy algorithm as the stationary measure is not known!

Critic in Actor/Critic

$$Q_{\boldsymbol{w}} \simeq q_{\pi_\theta}$$

Critic

- On-line TD learning with interaction following π_θ.
- Off-Policy TD learning is possible if the policy used for any action is stored.
- Approximate off-policy TD learning is possible using a replay buffer providing π_θ is changing slowly.

- May lead to 3 scales algorithm (Actor/Critic Target/Critic)
- As mentionned in the previous slide, much harder to do off-line update for the actor.

Off-Line Actor

$$J'_\mu(\pi) = \sum_s \mu(s) v_\pi(s)$$

Off-Line Actor

- Idea proposed in 2012.
- Key lemma in the paper

$$\nabla J'_\mu(\pi_\theta) \simeq \sum_s \mu(s) \sum_a \pi_\theta(a|s) \nabla \pi_\theta(a|s) q_{\pi_\theta}(s,a)$$

  turns out to be wrong!
- Still used as a heuristic justification of many algorithms!
- Explicit formula for $\nabla J'_\mu(\pi_\theta)$ can be obtained but much harder to use...

Outline

PPO: Minorize-Majorization Algorithm

$$J_{\mu_0}(\pi') \geq J_{\mu_0}(\pi) + \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(s|a) - \pi(s|a))\, a_\pi(s,a) \right) - \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Ideal Minorize-Majorization Algorithm

- At step k, find θ_{k+1} maximizing
$$J_{\mu_0}(\pi_\theta|\pi_{\theta_k}) = \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right) - \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \max_{s,a} |a_{\pi_{\theta_k}}(s,a)|$$
- By construction, $J_{\mu_0}(\pi_{\theta_{k+1}}) \geq J_{\mu_0}(\pi_{\theta_k})$

- Sample efficient algorithm as the same trajectory can be (re)used in the optimization.

PPO: Optimization

$$J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) \, a_{\pi_{\theta_k}}(s,a) \right)$$

$$- \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \max_{s,a} |a_{\pi_{\theta_k}}(s,a)|$$

Optimization

- Gradient descent is possible.
- Gradient of the first term can be approximated using a critic by

$$\sum_s \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a \pi_\theta \nabla \pi_\theta(s|a) A_{\pi_{\theta_k}}(s,a) \right)$$

- Gradient of the second term more involved.

- Simpler (TRPO like) strategy: optimize

$$\sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a)) \, a_{\pi_{\theta_k}}(s,a) \right)$$

under $\max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \leq \epsilon$ and reduce ϵ there is no gain.

PPO: KL Relaxation

$$J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right)$$
$$- \frac{2\gamma R_{\max}}{(1-\gamma)^2} \max_s \mathrm{KL}(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))$$

TRPO/PPO Optimization

- Replace the ℓ_1 norm by a KL divergence.
- In practice, replace the max by an average and replace $\frac{2\gamma R_{\max}}{(1-\gamma)^3}$ by parameter β and replace the a_{π_k} by an estimate A_{π_k}.
- PPO: Gradient descent of the relaxed goal.
- TRPO: Constrained optimization.

- Adaptive scheme to set β.
- Can be used with continuous action.

PPO: Clipped Objective

$$\sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a \pi_{\theta_k}(s|a) \min\left(\frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s,a), \mathrm{clip}(1-\epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)}, 1+\epsilon) a_{\pi_{\theta_k}}(s,a) \right) \right)$$

Clipped Objective

- Insight by (re)substracting $\sum_a \pi_{\theta_k}(s|a) a_{\theta_k}(s,a) = 0$:

$$\sum_a \min\left((\pi_\theta(s|a) - \pi_{\theta_k}(s,a))\, a_{\pi_{\theta_k}}(s,a), \mathrm{clip}(-\epsilon, \pi_\theta(s|a) - \pi_{\theta_k}(s,a), \epsilon) a_{\pi_{\theta_k}}(s,a) \right)$$

$$= \sum_a \mathrm{clip}(-\epsilon \pi_{\theta_k}(s,a), \pi_\theta(s|a) - \pi_{\theta_k}(s,a), \epsilon \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a)$$

$$- \max\left(0, -(\pi_\theta(s|a) - \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a) - \epsilon \pi_{\theta_k}(s,a) |a_{\pi_{\theta_k}}(s,a)| \right)$$

- First term amount to replace π_θ by a policy

$$\tilde{\pi}_\theta(a|s) = \mathrm{clip}(\pi_{\theta_k}(a|s)(1-\epsilon), \pi_\theta(a|s), \pi_{\theta_k}(a|s)(1+\epsilon)) + \eta_s \pi_{\theta_k}(a|s)$$

  where η is so that $\tilde{\pi}$ is a probability for all s and $\|\tilde{\pi}_\theta(\cdot,s) - \pi_{\theta_k}(\cdot,s)\|_1 \leq \epsilon$
- Second term: hinge loss type penalization of policy π_θ penalizing *bad* actions.

- Very efficient for discrete actions.

Discounted

PPO: Stationary Objective

$$\sum_{s,t} \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right) - \beta \max_s \mathrm{KL}(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))$$

$$\sum_{s,t} \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a \pi_{\theta_k}(s|a) \min \left(\frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s,a), \mathrm{clip}(1-\epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)}, 1+\epsilon) a_{\pi_{\theta_k}}(s,a) \right) \right)$$

Stationary Objective

- Amount to replace $J_{\mu_0}(\pi)$ by $J_{\mu_\pi}(\pi)$
- Most common implementation of PPO...
- But no way to justify it mathematically!
- May explain the (possible) instabilities of PPO.

- More on this later...

DPG: Deterministic Policy Gradient

$$J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \quad \text{with deterministic policy } \pi_\theta(a|s) = \mathbf{1}_{a=h_\theta(s)}$$

$$\nabla J_{\mu_0}(\pi_\theta) = \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \nabla_a q(S_t, h_\theta(S_t)) \nabla h_\theta(S_t)$$

Deterministic Policy Gradient

- Deterministic policy replaced by a randomized one centered on $h_{\theta(s)}$ in the interactions!
- Critic trained with a TD variant of DQN.
- Same formula by using a policy $\pi_\theta = \mathrm{N}(h_\theta(s), \sigma^2 \mathrm{Id})$ and letting σ goes to 0.
- Off-Policy as claimed?
- Yes for the actor but no theoretical justification for the critic!
- In practice, the buffer contains only samples using a policy close to the current one. . .

SAC: A New Goal

$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$

A Modified Reward

- Modification of the reward to favor high entropy policy:
$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$
- Goal:
$$J(\pi) = \mathbb{E}_\pi \left[\sum_t \gamma^t \left(R_t + \lambda \mathcal{H}(\pi(S_t))\right) \right]$$
- Soft value function implicitly defined as the fixed point of
$$\mathcal{T}^\pi q_\pi(s,a) = r_\pi(s,a) + \gamma \sum_{s'} p(s'|s,a) v_\pi(s')$$
$$\text{where} \quad v_\pi(s,a) = \sum_a \pi(a|s) \left(q_\pi(s,a) - \lambda \log \pi(a|s)\right)$$

SAC: Policy Improvement and Optimal Policy

$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$

A Modified Policy Improvement Lemma

- Policy improvement rule:

$$\pi^+(\cdot|s) = \underset{\pi(\cdot|s)}{\operatorname{argmax}} \sum_a \pi(a|s) \left(q(s,a) - \lambda \log(\pi(a|s))\right)$$

$$\pi^+(a|s) \propto \exp(\frac{1}{\lambda} q(s,a))$$

  implies $G_{\pi^+}(s,a) \geq G_\pi(s,a)$.

- At convergence, $J(\pi^\star)$ is optimal!
- Convergence in the finite setting.

SAC: Parametrization

$$\pi \sim \pi_\theta \quad \text{and} \quad q(s,a) \sim Q_{\boldsymbol{w}}$$

SAC Choices

- Fitted TD learning for Q:
$$\boldsymbol{w} \simeq \operatorname{argmin} \sum_{(S,A,R,S') \in \mathcal{B}} \left(R + \mathbb{E}_{\pi_\theta}\left[\gamma Q_{\overline{\boldsymbol{w}}}(S',a) - \lambda \log \pi_\theta(a|S')\right] - Q_{\boldsymbol{w}}(S,A)\right)^2$$
where the trajectory pieces are samples from a replay buffer and $\overline{\boldsymbol{w}}$ is a slowdown version of $\boldsymbol{w}$ (two-scales algorithm).
- Online version rather than batch...
- Fitted KL for π:
$$\theta \simeq \operatorname{argmin} \sum_{(S,A,R,S') \in \mathcal{B}} \mathrm{KL}(\pi_\theta(\cdot|S) | \exp -\lambda Q_{[}\overline{\boldsymbol{w}}](S,\dot{)}/Z_{\overline{\boldsymbol{w}}}(S))$$
$$\simeq \sum_{(S,A,R,S') \in \mathcal{B}} \mathbb{E}_{\pi_\theta}\left[\frac{1}{\lambda} \log \pi_\theta(a|S) - Q_\theta(a|s)\right]$$

Outline

Outline

Total Reward

$$v_\Pi(s) = \mathbb{E}_\Pi\left[\sum_{t'=1}^{+\infty} R_{t+1} \middle| S_0 = s\right]$$

- Total reward not necessarily well defined!
- Need to **assume** this is the case!

Properness Assumptions - Finite duration of episodes

- H-proper policy: It exists an absorbing state s_{abs} such that $\forall s,\ \mathbb{E}_\Pi[\min_{t, S_t = s_{\text{abs}}} t | S_0 = s] \leq H < +\infty$
- Episodic model: every policy is H-proper $\sim$ discounted setting for a weighted sup-norm.
- Stochastic Shortest Path: there is a proper policy and any non proper policy Π is such that $\exists s, v_\Pi(s) = -\infty$.

- Other models proposed by Puterman (Positive Bounded and Negative Models) have been abandoned by Puterman himself!

Bellman Operator and Optimality Equation

$$\sup_{\Pi} v_{\Pi}(s) = v_{\star}(s) = \underbrace{\max_a r(s,a) + \sum_{s'} p(s'|s,a) v_{\star}(s')}_{\mathcal{T}^{\star}(v_{\star})(s)}$$

- Similar to the discounted setting as:
  - We can focus on Markovian policy.
  - The optimal value $v_{\star}$ satisfies the Bellman optimality equation.

But...

- $\mathcal{T}^{\star}$ is not a contraction and thus there may be several solutions of the equation.
- If π is such that $\mathcal{T}^{\pi} v_{\star} = \mathcal{T}^{\star} v_{\star}$, we need to assume that $\limsup (P^{\pi})^n v_{\star}(s) \leq 0$ to prove that $\Pi = (\pi, \pi, \ldots)$ is optimal.
- There may not exist an optimal policy!

- Existence of optimal policies in the finite state-action setting by defining the total reward to the limit of discounted setting when $\gamma \to 1$ and using the finiteness of the policy set...

Stochastic Shortest Path

$$\Pi\ H\text{-proper} \Leftrightarrow \forall s,\ \mathbb{E}_\Pi\left[\min_{t, S_t = s_{\text{abs}}} t \middle| S_0 = s\right] \leq H < +\infty$$

Assumptions

- It exists a proper policy.
- For any improper policy, it exists s such that $v_\Pi(s) = -\infty$.

Properties

- For any proper policy, v_π is the unique solution of $v = \mathcal{T}^\pi v$, and $\mathcal{T}^\pi$ is a contraction.
- $v_\star$ is the unique solution of $v = \mathcal{T}^\star v$.
- Value Iteration and Policy Iteration converge in a stable manner.
- Modified Policy Iteration converges provided $v_0 \leq \mathcal{T}^\star v_0$.

$$\delta_t = R_t + Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

Prediction

- Convergence of TD-learning algorithms for any proper policy.

$$\delta_t = R_t + \max_Q(S_{t+1}, a) - Q(S_t, A_t)$$

Planning

- Convergence of Q-learning algorithms is the Stochastic Shortest Path setting if the Q estimates remain bounded.

- See *Neuro-Dynamic Programming* from Bertsekas and Tsitsiklis!
- May be very slow in practice!

Stochastic Shortest Path and Policy Gradient

$$\nabla v_{\pi_\theta}(s) = \sum_{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) a_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s]$$

$$= \sum_s \left(\sum_t \mathbb{P}_{\pi_\theta}(S_t = s|S_0 = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right)$$

Policy Gradient

- Formula valid in the Stochastic Shortest Path Assumption (if the current policy is proper).
- Approximate Policy Improvement Lemma with a H^2 multiplicative constant (instead of $O(H)$).

Actor-Critic

- Valid approach provided all the policies considered remain propers.
- Main difficulty is to maintain a good estimate of $q_{\pi_\theta}\dots$

Outline

Average Return

$$\overline{v}_\Pi(s) = \lim_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s) = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_\Pi\left[\sum_{t=1}^{T} R_t \middle| S_0 = s\right]$$

$$\longrightarrow \overline{v}_{+,\Pi}(s) = \limsup_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s)$$

$$\overline{v}_{-,\Pi}(s) = \liminf_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s)$$

Average Return(s)

- Limit $\overline{v}_\Pi$ may not be defined!
- **Prop:** $\overline{v}_\Pi$ is well defined if Π is stationary and $\frac{1}{T}\sum_{t=1}^{T}(P^\pi)^{t-1}$ tends to a stochastic matrix.
- Limits $\overline{v}_{+,\Pi}$ and $\overline{v}_{-,\Pi}$ always defined!

Average Returns and Optimality

$$\overline{v}_{+,\star}(s) = \sup_{\Pi} \overline{v}_{+,\Pi}(s) \quad \text{and} \quad \overline{v}_{-,\star}(s) = \sup_{\Pi} \overline{v}_{-,\Pi}(s)$$

Optimality of $\Pi_\star$

- Average optimal:

$$\overline{v}_{-,\Pi_\star} \geq \overline{v}_{+,\star}(s)$$

- Lim-sup average optimal (best case analysis):

$$\overline{v}_{+,\Pi_\star} \geq \overline{v}_{+,\star}(s)$$

- Lim-inf average optimal (worst case analysis):

$$\overline{v}_{-,\Pi_\star} \geq \overline{v}_{-,\star}(s)$$

- More complex setting!
- Let's start with Prediction...

Prediction for a Stationary Markov Policy

$$\overline{v}_\Pi(s) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P_\pi^{t-1} r_\pi = \left(\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P_\pi^{t-1} \right) r_\pi = P_\pi^\infty r_\pi$$

Stochastic Matrix P_π^∞

- Measures the average amount of time spend on a state s' starting from state s at $t = 0$ when using policy π.
- Structure linked to the properties of the resulting Markov chain:
  - If aperiodic, $P_\pi^\infty = \lim_T P_\pi^T$ i.e. P_π^∞ is close to the probability of reaching s' from s at any large T.
  - If unichain, then P_π^∞ has identical rows and corresponds to the stationary distribution.
  - If multichhain, then P_π^∞ has a diagonal block structure with rows equal withing each block corresponding to the stationary distribution in each chain.

- Implies that $\overline{v}_\Pi(s) = \overline{v}_\Pi(s')$ in the Markov process is unichain.
- Limit P_π^∞ may be hard to compute...

Average Reward and Relative Value Functions

$$U_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=1}^{\infty}(R_t - \overline{v}_\pi(S_t))\middle| S_0 = s\right] \quad \Leftrightarrow U_\pi = \underbrace{(\mathrm{Id} - P_\pi + P_\pi^\infty)^{-1}(\mathrm{Id} - P_\pi^\infty)}_{H_\pi} r_\pi$$

Link between U_π and $\overline{v}_\pi$

- $(\mathrm{Id} - P_\pi)\overline{v}_\pi = 0$
- $\overline{v}_\pi + (I - P_\pi)U_\pi = r_\pi$

Characterization by a system

- If $(\mathrm{Id} - P_\pi)\overline{v} = 0$ and $\overline{v} + (I - P_\pi)U = r_\pi$ then
  - $\overline{v} = \overline{v}_\pi$,
  - $U = U_\pi + u$ with $(I - P_\pi)u = 0$,
  - If $P_\pi^\infty U = 0$ then $u = 0$.

- Prediction possible by solving this system as we do not need U_π.

Optimality Equations

$$\overline{v}(s) = \max_a \sum_{s'} p(s'|s,a)\overline{v}(s')$$

$$U(s) + \overline{v}(s) = \max_{a \in B_s} r(s,a) + \sum_{s'} p(s'|s,a)U(s) \text{with } B_s = \{a | \sum_{s'} p(s'|s,a)\overline{v}(s') = \overline{v}(s)\}$$

$$\pi_\star(s) \in \operatorname{argmax}_{a \in B_s} r(s,a) + \sum_{s'} p(s'|s,a)U(s)$$

Existence

- If there is a solution $(\overline{v}, U)$ of the system then $\overline{v} = \overline{v}_\star$ and $\pi_\star$ is an optimal policy.
- There may exist other optimal policies not satisfying the argmax property.
- There may not exist solutions to the system.

- Associated relative value iteration and modified policy iteration can be defined.
- Convergence under strong assumptions...

Average Return and Relative Value Functions

$$r(\pi) = \lim_T \mathbb{E}_\pi\left[\frac{1}{T}\sum_{t=0}^{T-1} R_t\right] = \sum_s \mu_\pi(s) \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) r$$

$$G_t = \sum_{t' \geq t} (R_t - r(\pi))$$

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s] \quad \text{and} \quad q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

Connection with Stochastic Shortest Path

- Provided there is a state s that is visited with positive probability in the first m steps for any starting state and any policy.
- $r(\pi)$ is the average cost between a visit s and the next one...

Reinforcement Learning Algorithms

- Simultaneous estimation of q and r...
- Much less theory as there is no contraction!

Algorithm(s)

Average: Planning by SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t = 0$, $r = 0$
Pick initial state S_0 following μ_0
repeat
- $N(S_t) \leftarrow N(S_t) + 1$
- Pick action A_t according to $\pi(\cdot|S_t)$
- $Q(S_{t-1}, A_{t-1}) \leftarrow Q(S_{t-1}, A_{t-1}) + \alpha(N(S_{t-1}, A_{t-1}))\,(R_t - r_{t-1} + \gamma Q(S_t, A_t) - Q(S_{t-1}, A_{t-1}))$
- $r \leftarrow r + \alpha_t(R_t - r)$
- $\Pi(S_{t-1}) = \operatorname{argmax}_a Q(S_{t-1}, a)$ (plus exploration)
- $t \leftarrow t + 1$

until $t == T$
output: Deterministic policy $\tilde{\pi}(s) = \operatorname{argmax}_a Q(s,a)$

- Q-learning variant (known as R-learning) and other estimations of r exist.
- No convergence proof.

Policy Gradient

$$\nabla r(\pi) = \lim_T \frac{1}{T} \mathbb{E}_\pi \left[\sum_{i=1}^{T} \nabla \log \pi(A_t|S_t) q_\pi(S_t, A_t) \right]$$

$$\nabla r(\pi) = \lim_T \frac{1}{T} \mathbb{E}_\pi \left[\sum_{i=1}^{T} \nabla \log \pi(A_t|S_t) a_\pi(S_t, A_t) \right]$$

Policy Gradient

- REINFORCE type algorithms, using MC estimate of q and a are possible,
- but q and a are the relative ones, not the classical ones, and are much harder to estimate.

- Actor/Critic algorithms combining parametric estimation of q (or a) and gradient exist.

Outline

To Discount or Not?

To Discount: $J(\pi) = \mathbb{E}_\pi\left[\sum_t \rho^t R_t\right]$ $\quad Q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_t \rho^t R_t \middle| s_0 = s, a_0 = a\right]$

or Not (SSP): $J(\pi) = \mathbb{E}_\pi\left[\sum_t R_t\right]$ $\quad Q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_t R_t \middle| s_0 = s, a_0 = a\right]$

To Discount or Not? Open Question!

- Discount is (quite) artificial.
- No discount in the evaluation part most of the time.
- Discount often used in training due to better convergence for value functions. . . toward a (quite) artificial policy target!

- In practice, often hybrid scheme with no discount for the policy gradient part, but discount for the value functions part! No strong justification but often better numerical performance!
- Average reward much less used!

Outline

POMDP

$$o \sim \mathbb{P}(\cdot|s, a)$$

Partially Observed Markov Decision Process

- MDP strongest assumption is that s is observed!
- POMDP replaces this assumption by the observation of o with a known law of $\mathbb{P}(o|s, a)$.
- Can be recasted as a MDP where the state is the probability of being in a state s given the current observation!
- Much higher dimensional setting!

- Policy gradient algorithms remain valid in the POMDP setting when replacing s with o.
- Difficult part is to obtain a good value function estimate.

Outline

Imitation Learning

$$\text{Good } S_t, A_t, (R_{t+1},) S_{t+1}, A_{t+1} \to \pi$$

$$\operatorname*{argmin}_\theta \sum_{i=1}^{t} \log \pi_\theta(A_t | S_t)$$

Imitation Learning

- Learn policy from demonstrations (observations).
- Most classical approach: maximum likelihood.
- Need to cover all states (possibly through the approximation)
- Reward is not used.

- DAGGER: Sequential approach to add feedback from trajectory with an estimated policy through the decision that would have been made.

Inverse Reinforcement Learning

$$\text{Good } S_t, A_t, S_{t+1}, A_{t+1} \text{ or } \pi \to R \to \pi^\star$$

Inverse Reinforcement Learning

- **Heuristic:** Learn a reward which **explains** the observed policy and used it to obtain a better policy (or to generalize to different models).
- No clear mathematical formulation:
  - Reward so that the observed policy is optimal (with a margin).
  - Expected return/optimal value function linked to observed policy (trajectories) probability (with entropic regularization)
  - Most generic formulation?

$$\min_{\pi'} \max_{R} \mathbb{E}_\pi[R] - \mathbb{E}_{\pi'}[R] + K(\pi') - C(R)$$

- Exact problem considered not always clear for a given algorithm (and different from one algorithm to another)!

- Very hard problem!

Learning from Preferences

$$S_t, A_t, S_{t+1}, A_{t+1} \text{ vs } S_t, A'_t, S'_{t+1}, A'_{t+1} \rightarrow R \rightarrow \pi^\star$$

Learning from Preferences

- Often easier to compare trajectories than to make a demonstration.
- **Reinforcement Learning from Human Feedback**: Learn a reward from the demonstration using a preference model (Bradley-Terry?) and use it to find a policy.
- **Direct Policy Optimization**: shortcut to optimize directly the policy thanks to the explicit preference model used.
- Proximity constrains are often added to avoid moving too fast from a current policy.

- Key to the performances of current LLMs.

LLM: Large Language Model

Outline

More!

- Regrets
- Sample optimality
- Robustness
- Multi-agents (Games. . .)
- LLM and world models. . .

Outline

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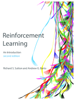

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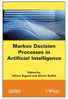

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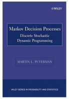

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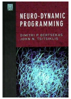

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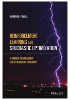

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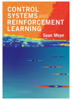

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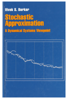

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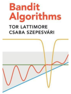

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Contributors

- Main contributor: E. Le Pennec
- Contributors: S. Boucheron, A. Dieuleveut, A.K. Fermin, S. Gadat, S. Gaiffas, A. Guilloux, Ch. Keribin, E. Matzner, M. Sangnier, E. Scornet.