Bandelets and Applications

Ch. Dossal, E. Le Pennec, S. Mallat, G. Peyré
CMAP (École Polytechnique) – Let It Wave – PMA (Université Paris 7)

Geometrical Image Representation

Geometrical Image Representation

- Signal processing requires to build sparse signal representations for compression, restoration, pattern recognition...

Geometrical Image Representation

- Signal processing requires to build sparse signal representations for compression, restoration, pattern recognition...
- Sparsity is derived from regularity.

Geometrical Image Representation

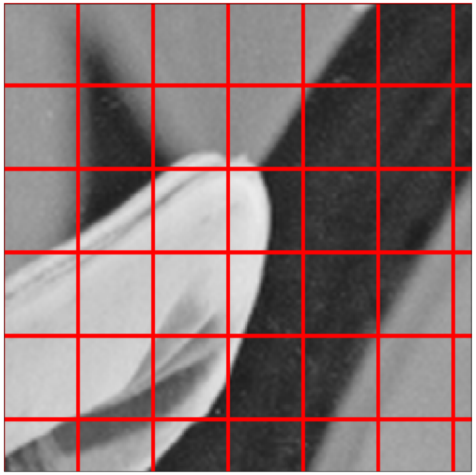
- Signal processing requires to build sparse signal representations for compression, restoration, pattern recognition...
- Sparsity is derived from regularity.
- Need to take advantage of geometrical image regularity to improve representations.

Geometrical Image Representation

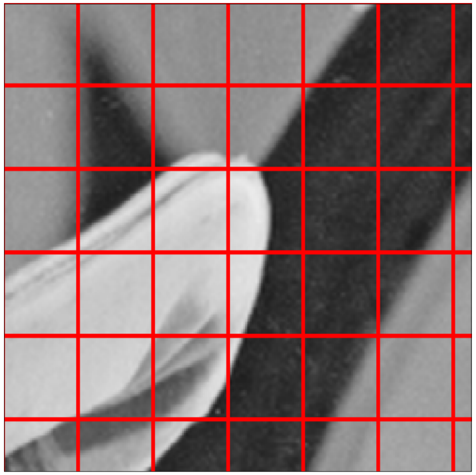
- Signal processing requires to build sparse signal representations for compression, restoration, pattern recognition...
- Sparsity is derived from regularity.
- Need to take advantage of geometrical image regularity to improve representations.
- Building harmonic analysis representations adapted to complex *geometry*.

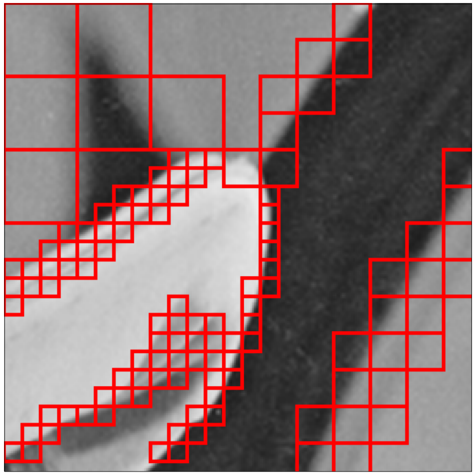
Transform Coding

Transform Coding

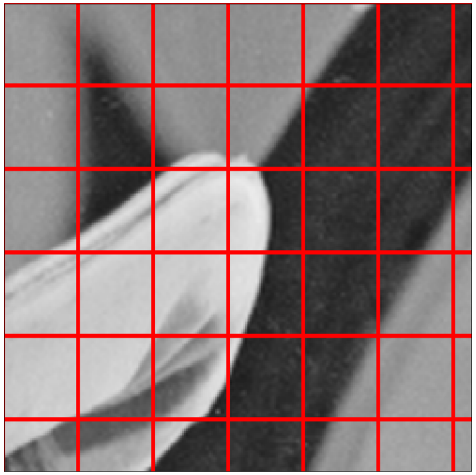
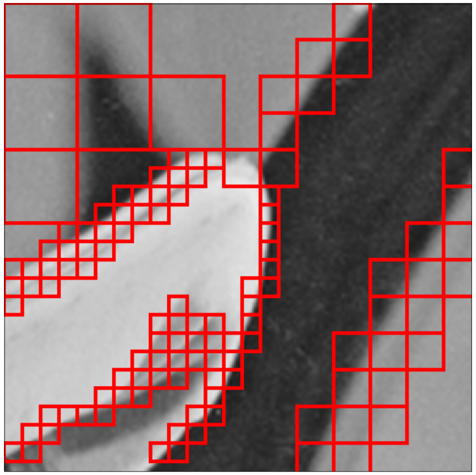


Transform Coding





Transform Coding

- JPEG: DCT (Transform) (80)
- JPEG 2000: Wavelet (Multiscale) (90)
- Now: Geometric Wavelets (Geometry) (??)

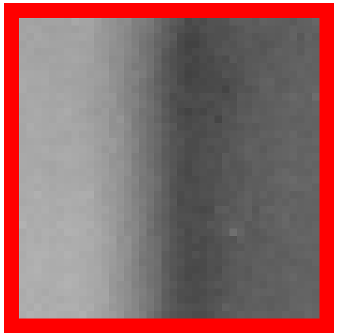
Edge Detection

Edge Detection

- An ill-posed problem.

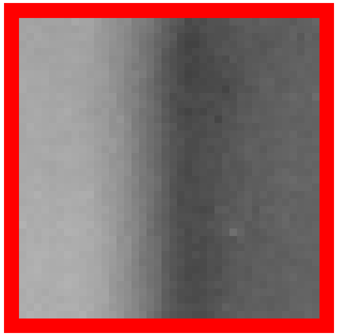
Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:



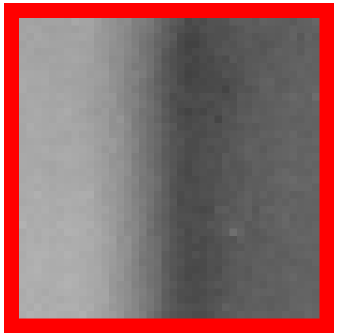
Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:



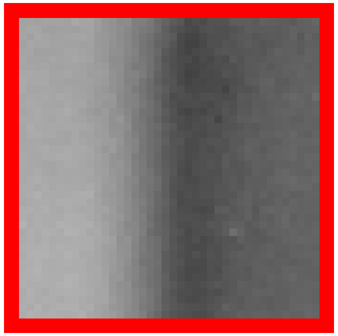
Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:

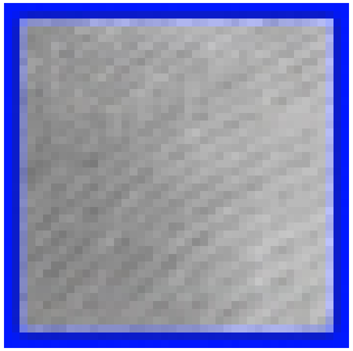


Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:

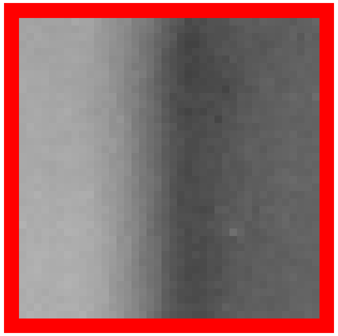


- Scale of geometric regularity:

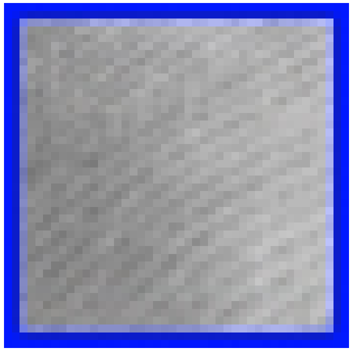


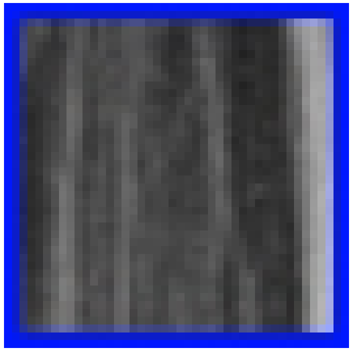
Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:



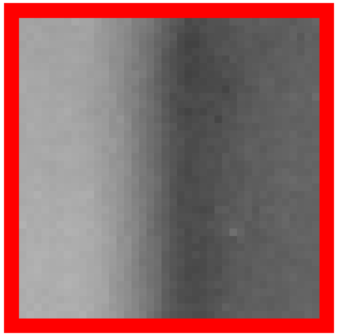
- Scale of geometric regularity:



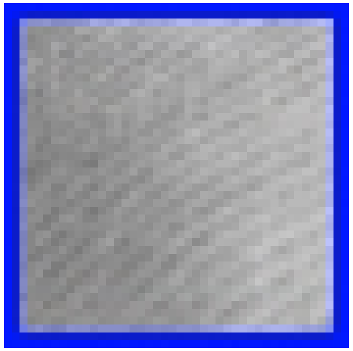
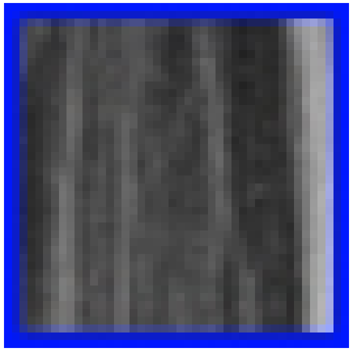
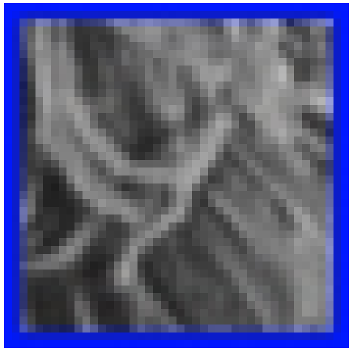


Edge Detection

- An ill-posed problem.
- Edges are blurred transitions:

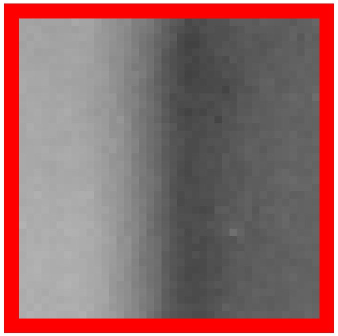



- Scale of geometric regularity:

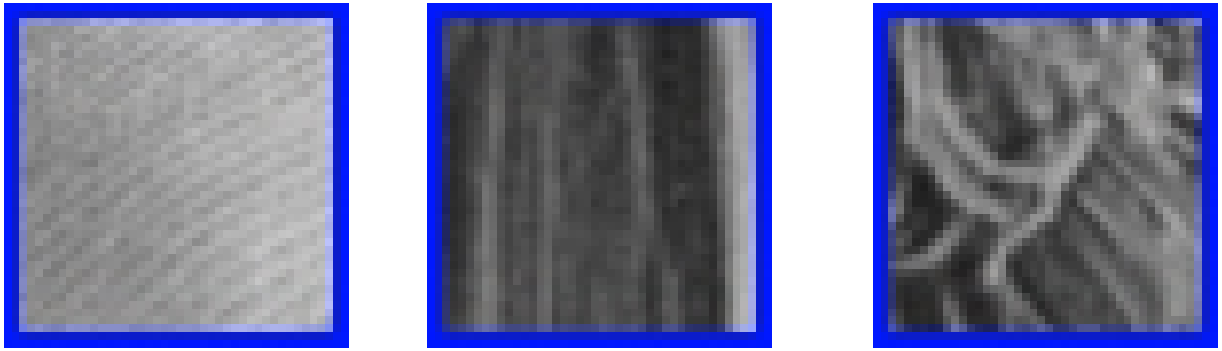




Edge Detection

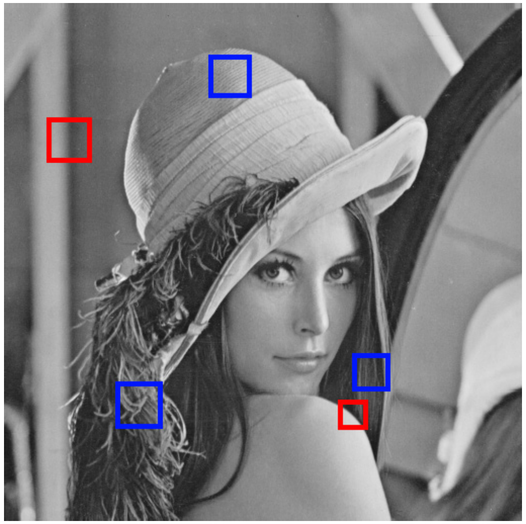
- An ill-posed problem.
- Edges are blurred transitions:



- Scale of geometric regularity:

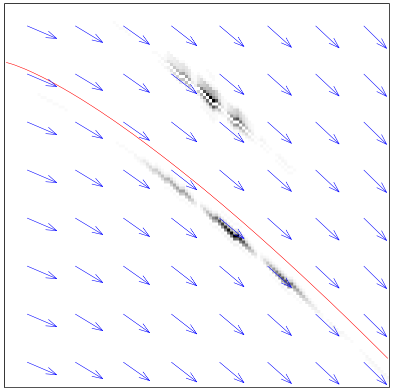


- How can the estimation of the geometry become well-posed ?

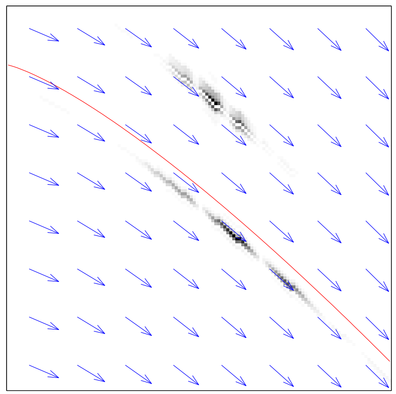


Bandelets

Bandelets



Bandelets

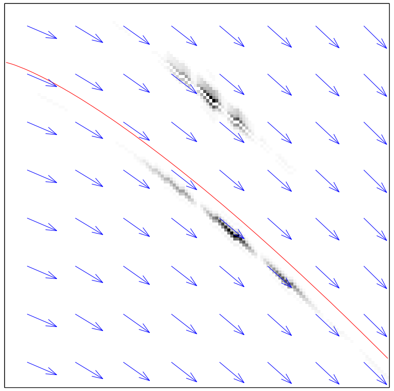


- Basis adapted to the geometry: bandelets with an anisotropic support that follows the direction of regularity of the image,

$$\left\{ \frac{1}{2^{(j+l)/2}} \Psi^d \left(\frac{x_1 - 2^l m_1}{2^l}, \frac{x_2 - c(x_1) - 2^j m_2}{2^j} \right) \right\}_{d,j,l \geqslant j, m_1, m_2} .$$

- Dyadic segmentation and associated geometry: bandelet basis adapted to an image.

Bandelets



- Basis adapted to the geometry: bandelets with an anisotropic support that follows the direction of regularity of the image,

$$\left\{ \frac{1}{2^{(j+l)/2}} \Psi^d \left(\frac{x_1 - 2^l m_1}{2^l}, \frac{x_2 - c(x_1) - 2^j m_2}{2^j} \right) \right\}_{d,j,l \geqslant j,m_1,m_2} .$$

- Dyadic segmentation and associated geometry: bandelet basis adapted to an image.
- Efficient optimization of this geometry: non linear approximation theorem.

$$\|f - f_M\|^2 \leqslant C M^{-\alpha}$$

Overview

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. Dossal)
  - Denoising
  - Deconvolution of seismic data

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. DOSSAL)
  - Denoising
  - Deconvolution of seismic data
- Session 4
  - Bandelets NG

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. Dossal)
  - Denoising
  - Deconvolution of seismic data
- Session 4
  - Bandelets NG

Sparse Representation in a Basis

Sparse Representation in a Basis

- Decomposition in an orthonormal basis $\mathbf{B} = \{g_m\}_{m\in\mathbb{N}}$

$$f = \sum_{m\in\mathbb{N}} \langle f, g_m \rangle \, g_m \ .$$

Sparse Representation in a Basis

- Decomposition in an orthonormal basis $\mathbf{B} = \{g_m\}_{m\in\mathbb{N}}$

$$f = \sum_{m\in\mathbb{N}} \langle f, g_m \rangle \, g_m \ .$$

- Approximation with M vectors chosen adaptively

$$f_M = \sum_{m\in I_M} \langle f, g_m \rangle \, g_m \, .$$

Sparse Representation in a Basis

- Decomposition in an orthonormal basis $\mathbf{B} = \{g_m\}_{m\in\mathbb{N}}$

$$f = \sum_{m\in\mathbb{N}} \langle f, g_m\rangle \, g_m \ .$$

- Approximation with M vectors chosen adaptively

$$f_M = \sum_{m\in I_M} \langle f, g_m\rangle \, g_m \, .$$

- To minimize $\quad \|f - f_M\|^2 = \sum_{m\notin I_M} |\langle f, g_m\rangle|^2,$

  select the M largest inner products:

$$I_M = \{m, |\langle f, g_m\rangle| > T_M\} \text{ : thresholding}$$

Sparse Representation in a Basis

- Decomposition in an orthonormal basis $\mathbf{B} = \{g_m\}_{m\in\mathbb{N}}$
$$f = \sum_{m\in\mathbb{N}} \langle f, g_m \rangle \, g_m \ .$$
- Approximation with M vectors chosen adaptively
$$f_M = \sum_{m\in I_M} \langle f, g_m \rangle \, g_m \, .$$
- To minimize $\quad \|f - f_M\|^2 = \sum_{m\notin I_M} |\langle f, g_m \rangle|^2,$

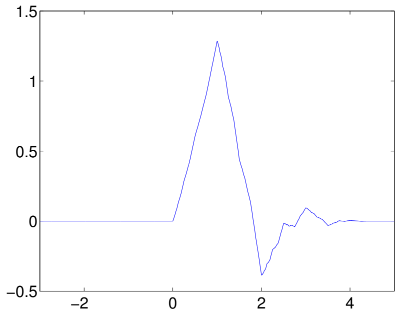
  select the M largest inner products:
$$I_M = \{m, |\langle f, g_m \rangle| > T_M\} \text{ : thresholding}$$
- **Problem:** Given that $f \in \Theta$, how to choose $\mathbf{B}$ so that
$$\|f - f_M\|^2 \leqslant C M^{-\beta} \ \text{ with } \beta \text{ large ?}$$

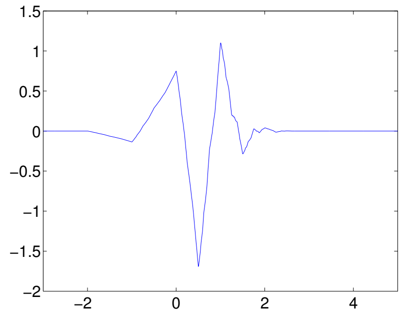
1D Wavelet Basis of $L^2[0, 1]$

1D Wavelet Basis of $L^2[0,1]$

- Constructed with a scaling function $\phi(x)$ and a mother wavelet $\psi(x)$

$\phi(x)$





$\psi(x)$

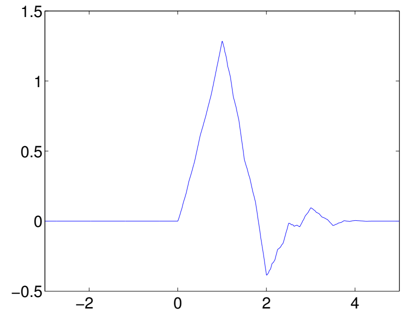
which are scaled by 2^j and translated by $2^j n$

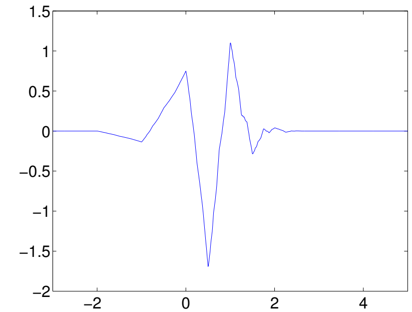
$$\phi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\phi\Big(\frac{x - 2^j n}{2^j}\Big) \quad , \quad \psi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\psi\Big(\frac{x - 2^j n}{2^j}\Big)$$

1D Wavelet Basis of $L^2[0,1]$

- Constructed with a scaling function $\phi(x)$ and a mother wavelet $\psi(x)$

$\phi(x)$



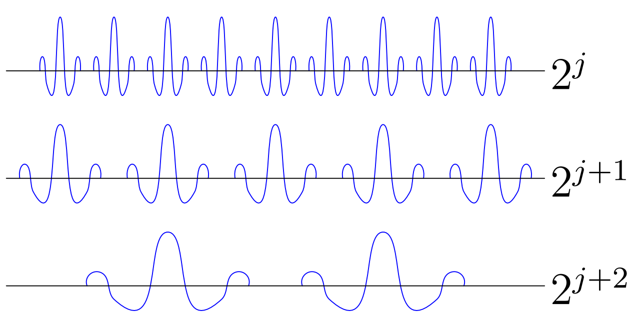


$\psi(x)$

which are scaled by 2^j and translated by $2^j n$

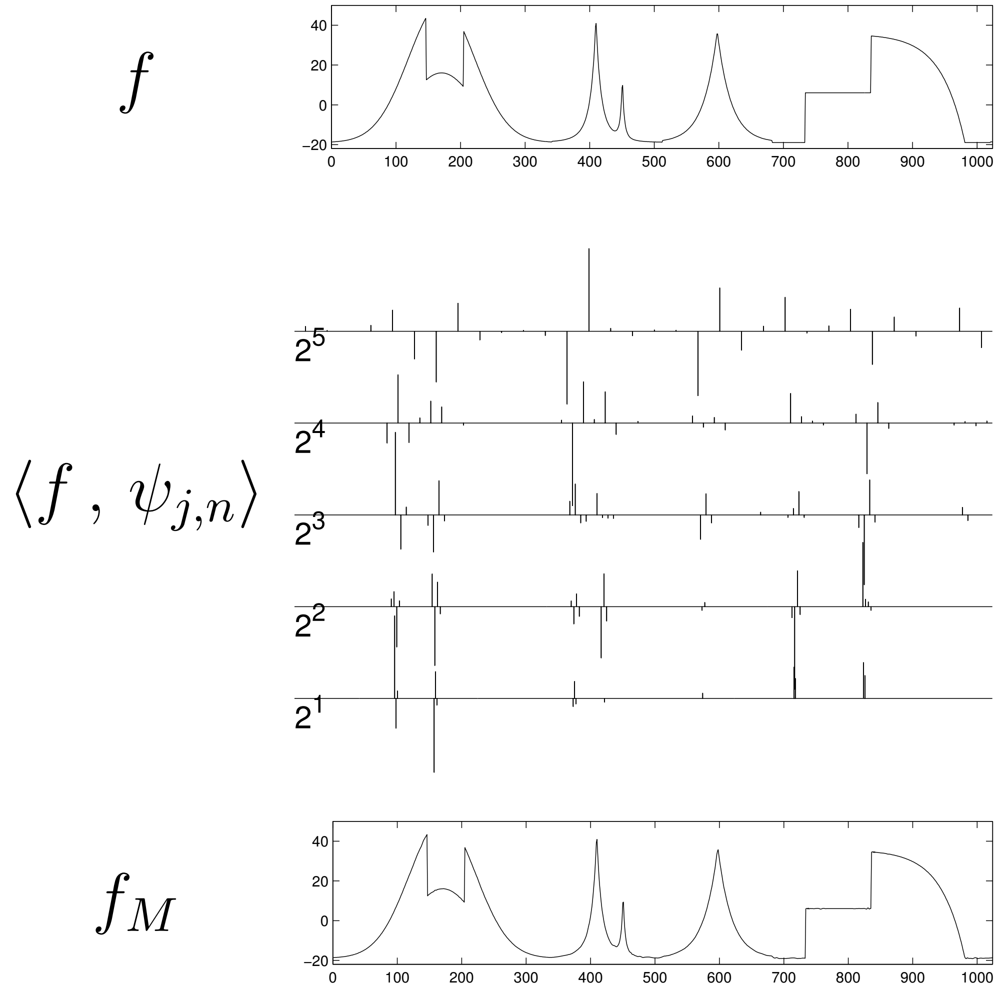
$$\phi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\,\phi\Big(\frac{x-2^j n}{2^j}\Big) \quad , \quad \psi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\,\psi\Big(\frac{x-2^j n}{2^j}\Big)$$

- $\mathbf{B} = \left\{\psi_{j,n}\right\}_{j\in\mathbb{N}\ ,\, 2^j n\in[0,1)}$ is an orthonormal basis of $L^2[0,1]$.



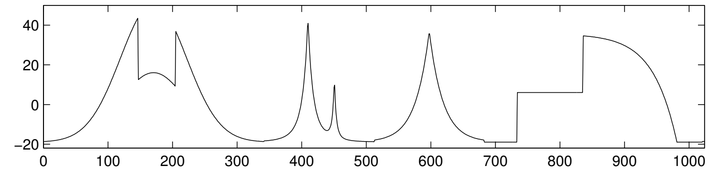
Non-Linear Approximation in a Wavelet Basis

Non-Linear Approximation in a Wavelet Basis

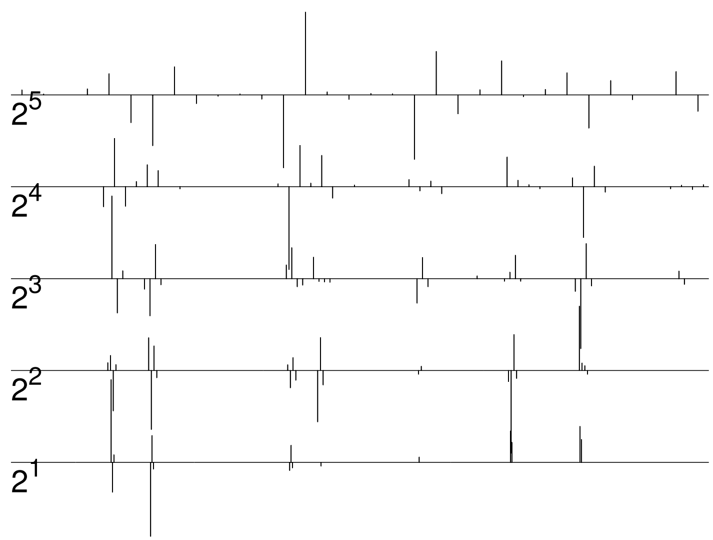


Non-Linear Approximation in a Wavelet Basis

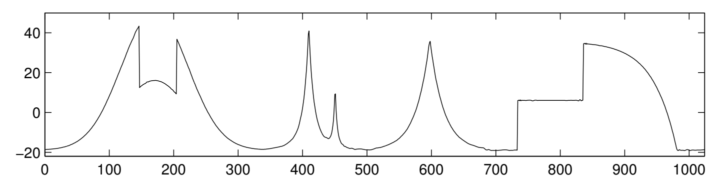
f



$\langle f , \psi_{j,n} \rangle$



f_M



- If f is piecewise $\mathbf{C}^{\alpha}$ and ψ has $p > \alpha$ vanishing moments then

$$\|f - f_M\|^2 \leqslant C\, M^{-2\alpha} \;.$$

Isotropic Separable 2D Wavelet Basis

Isotropic Separable 2D Wavelet Basis

- The family

$$\left\{ \begin{array}{ccc} \phi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2) & , & \psi_{j,n_1}(x_1)\,\phi_{j,n_2}(x_2) \\ & , & \psi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2) \end{array} \right\}_{(j,n_1,n_2)\in\mathbb{Z}^3}$$

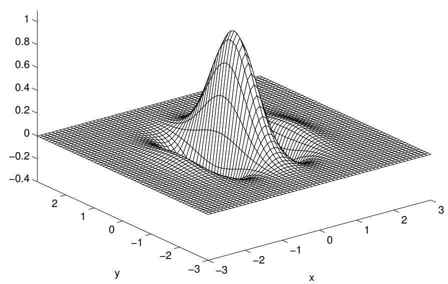
is an orthonormal basis of $L^2[0,1]^2$.

Isotropic Separable 2D Wavelet Basis

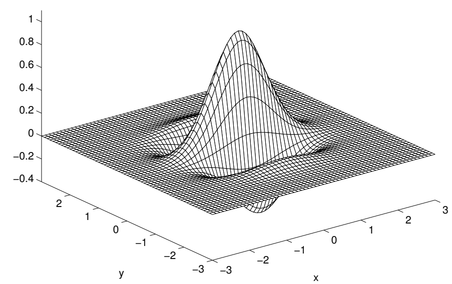
- The family

$$\left\{ \begin{array}{ccc} \phi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2) & , & \psi_{j,n_1}(x_1)\,\phi_{j,n_2}(x_2) \\ & , & \psi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2) \end{array} \right\}_{(j,n_1,n_2)\in\mathbb{Z}^3}$$

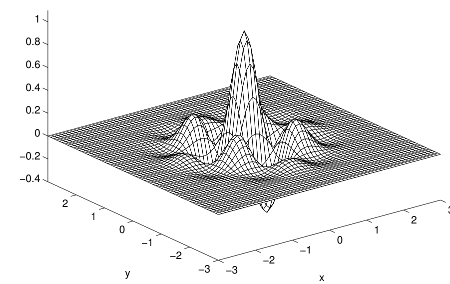
is an orthonormal basis of $L^2[0,1]^2$.



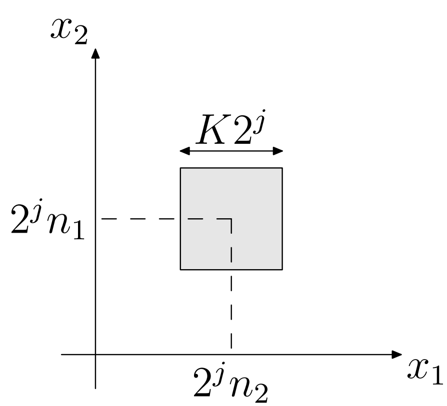
$\phi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2)$



$\psi_{j,n_1}(x_1)\,\phi_{j,n_2}(x_2)$



$\psi_{j,n_1}(x_1)\,\psi_{j,n_2}(x_2)$



Wavelets Support

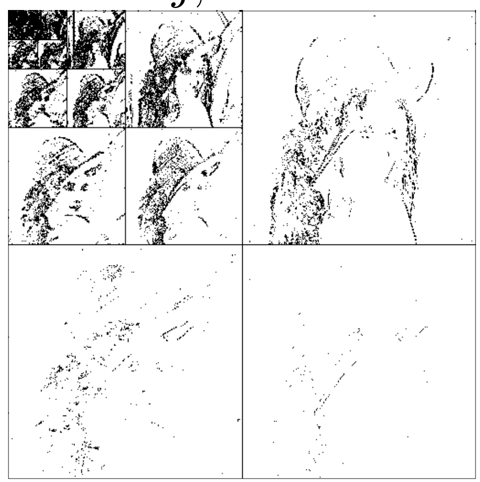
Successes and Failures of Wavelet Bases

Successes and Failures of Wavelet Bases

- Images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f

$|\langle f, \psi_{j,n}^{k} \rangle| > T_M$



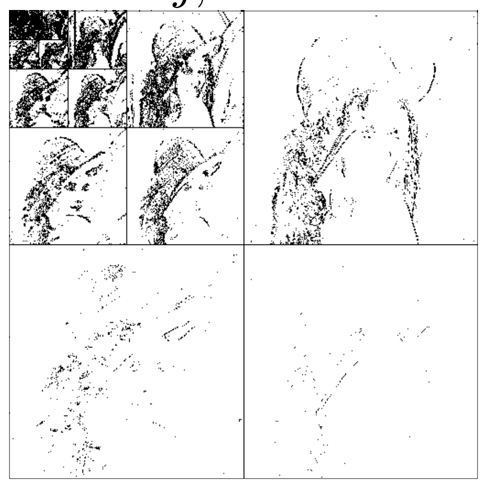
f_M

Successes and Failures of Wavelet Bases

- Images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f

$|\langle f, \psi_{j,n}^k \rangle| > T_M$



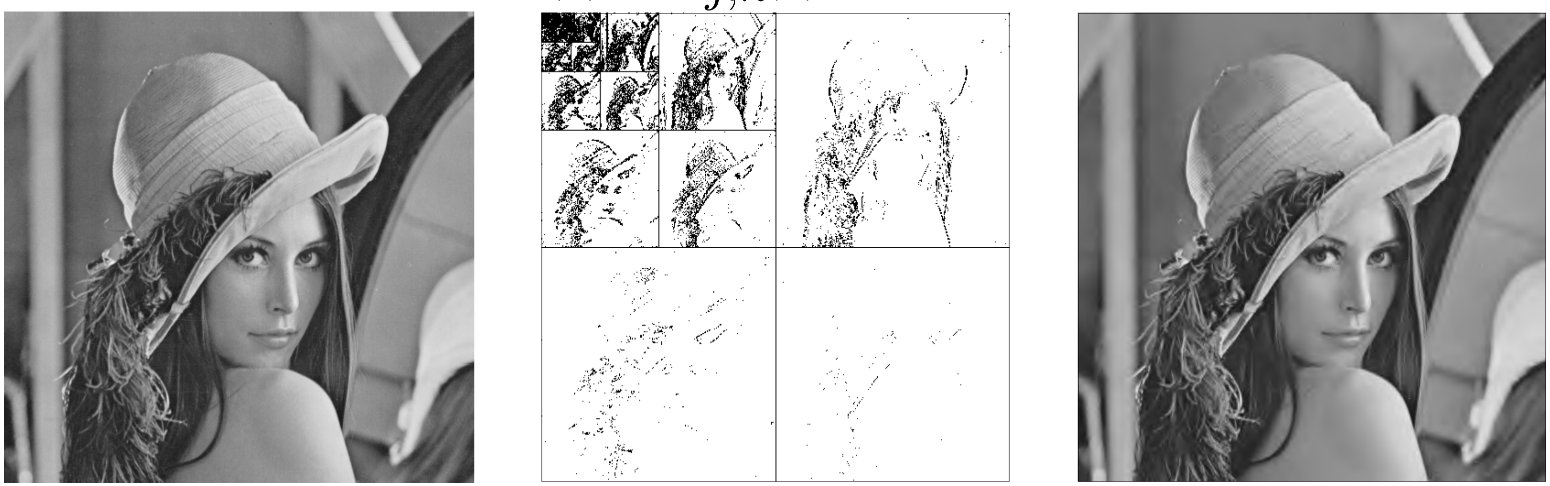
f_M

- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.

Successes and Failures of Wavelet Bases

- Images are decomposed in a two-dimensional wavelet basis and larger coefficients are kept (JPEG-2000).

f $\qquad$ $|\langle f, \psi_{j,n}^k \rangle| > T_M$ $\qquad$ f_M

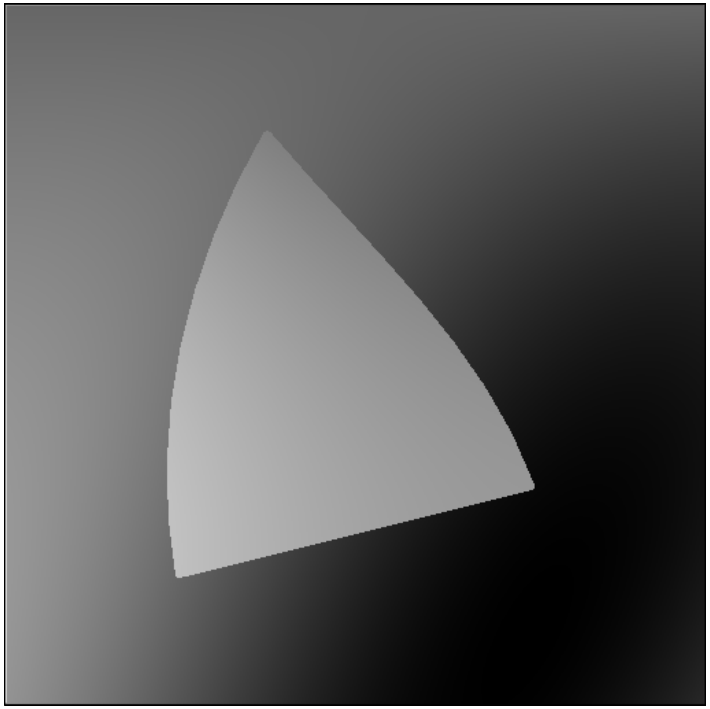


- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.
- But: does not take advantage of any geometric regularity.

Wavelet Approximation with Edges

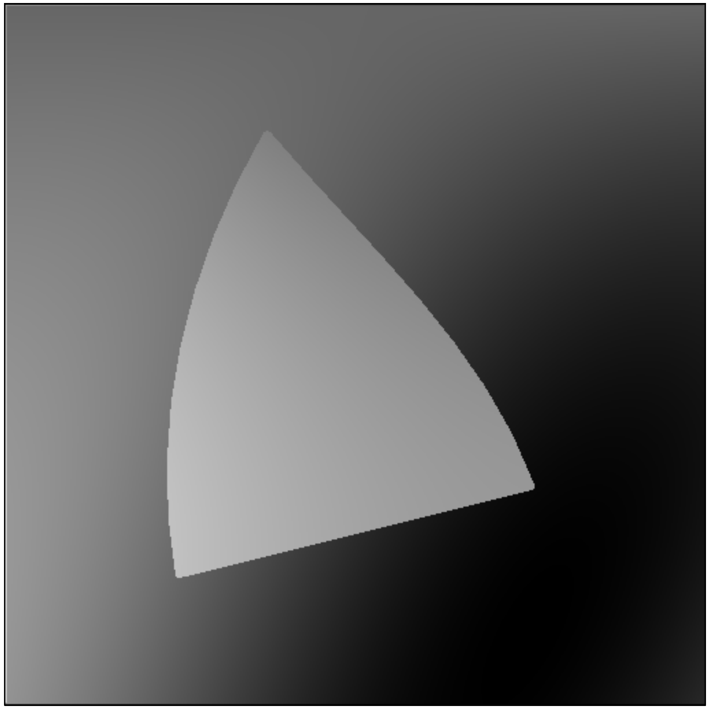
Wavelet Approximation with Edges

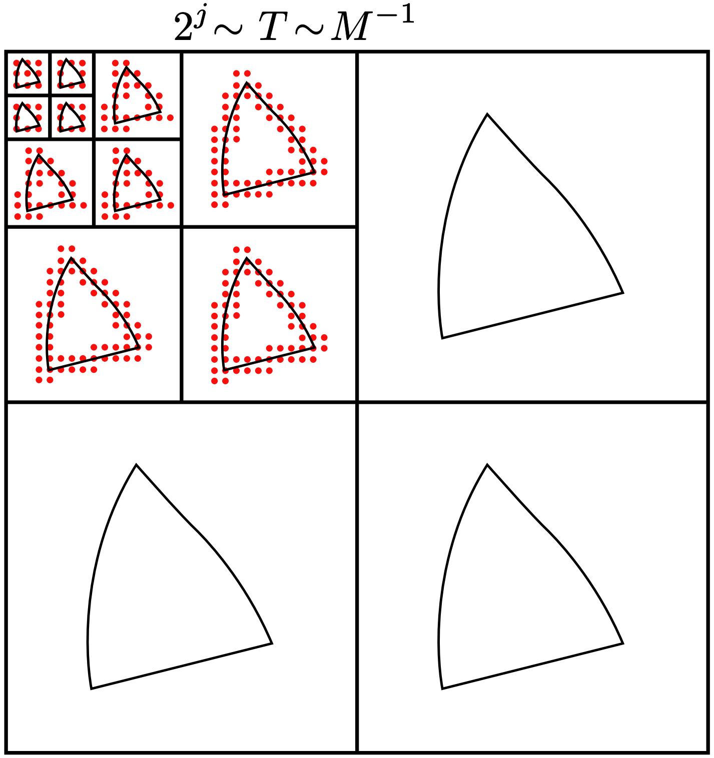
- Approximations of f which is $\mathbf{C}^{\alpha}$ away from $\mathbf{C}^{\alpha}$ "edge" curves:



Wavelet Approximation with Edges

- Approximations of f which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves:



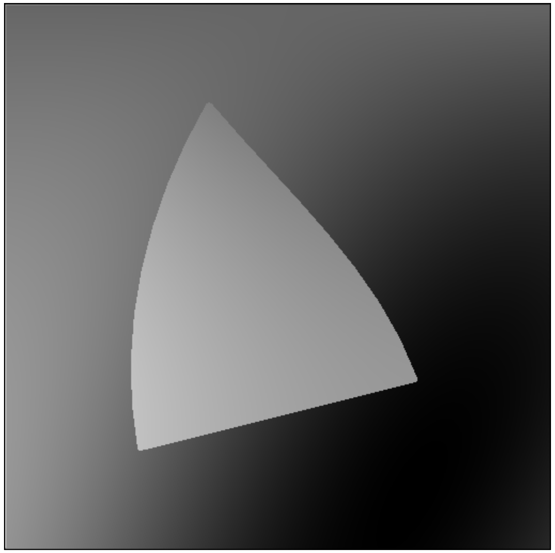


- with M wavelets: $\|f - f_M\|^2 \leqslant C\, M^{-1}$.

Geometric Finite Elements for Edges

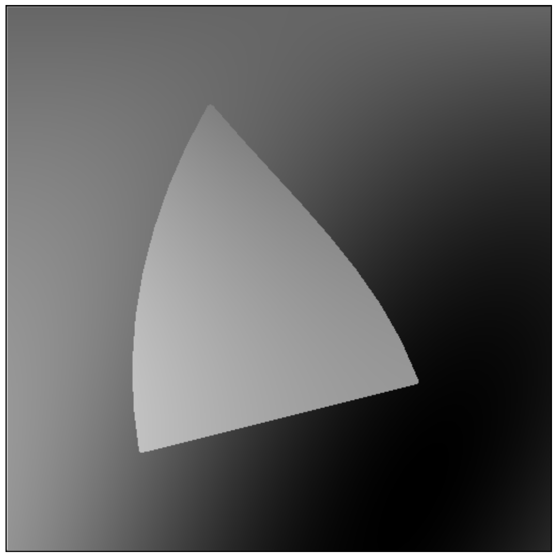
Geometric Finite Elements for Edges

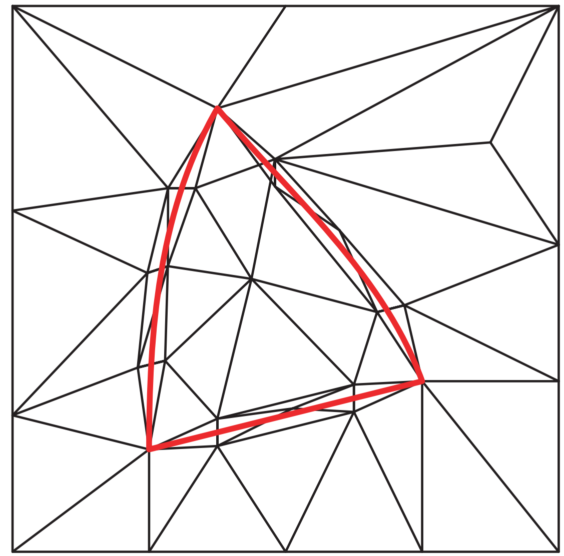
- Approximations of f which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves:



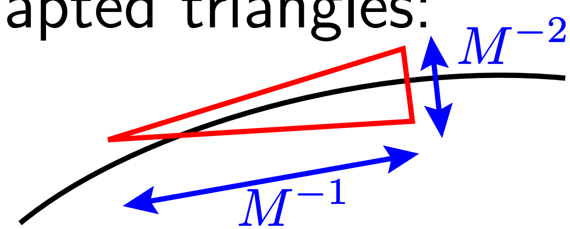
Geometric Finite Elements for Edges

- Approximations of f which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves:



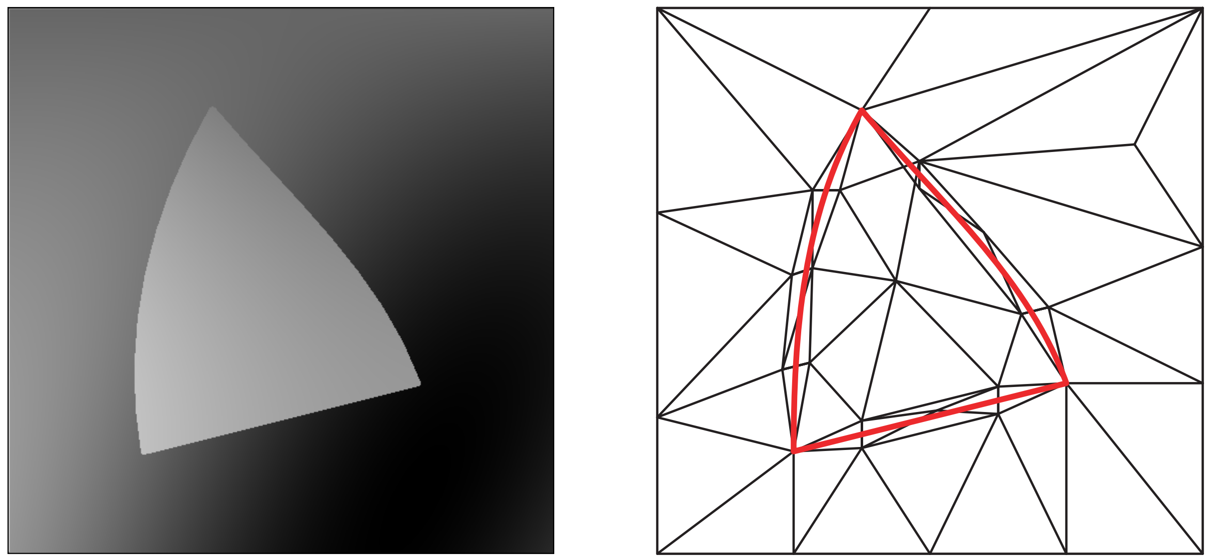


- Piecewise linear approximation over M adapted triangles: if $\alpha \geqslant 2$ then $\|f - f_M\|^2 \leqslant C\, M^{-2}$,

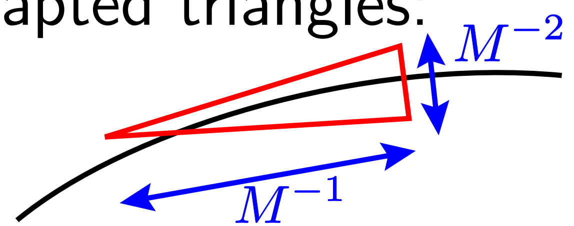


Geometric Finite Elements for Edges

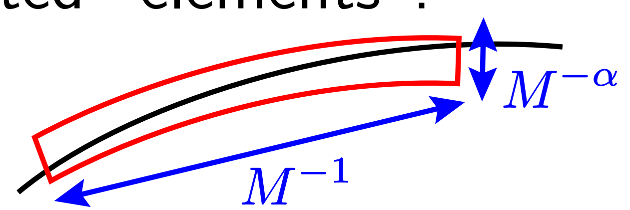
- Approximations of f which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves:



- Piecewise linear approximation over M adapted triangles: if $\alpha \geqslant 2$ then $\|f - f_M\|^2 \leqslant C\, M^{-2}$,



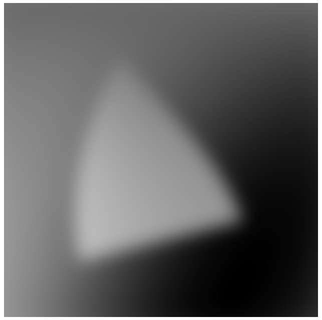
- Higher order approximation over M adapted "elements": $\|f - f_M\|^2 \leqslant C\, M^{-\alpha}$.



Adaptive Triangulation for Smooth Edges

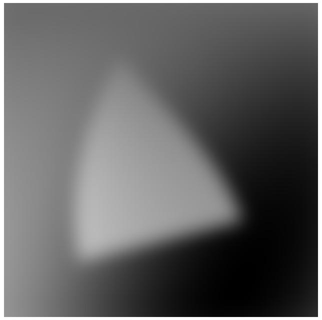
Adaptive Triangulation for Smooth Edges

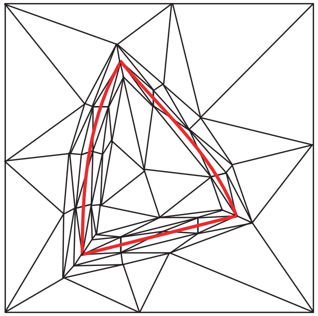
- Approximations of $f = \tilde{f} \star h_s$ which:
  - f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ “edge” curves ($\alpha \geqslant 2$):
  - h_s is a regularization kernel of size s



Adaptive Triangulation for Smooth Edges

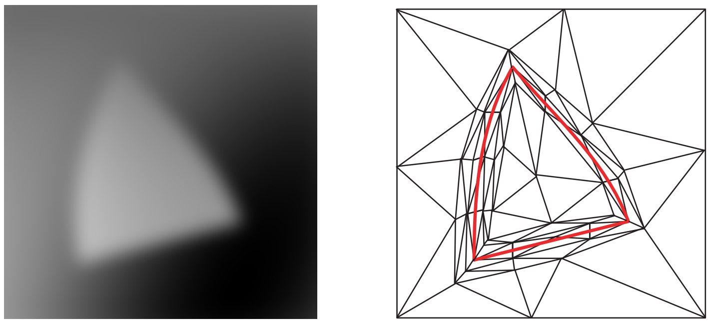
- Approximations of $f = \tilde{f} \star h_s$ which:
  - f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves ($\alpha \geqslant 2$):
  - h_s is a regularization kernel of size s



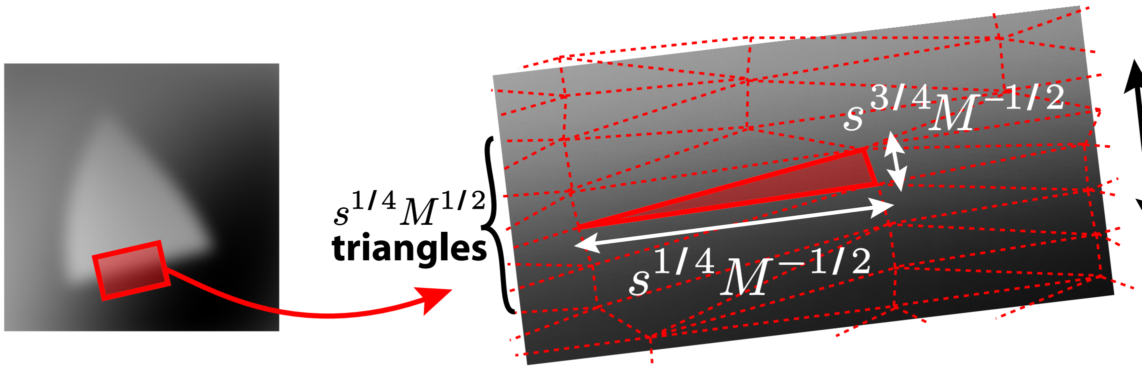


Adaptive Triangulation for Smooth Edges

- Approximations of $f = \tilde{f} \star h_s$ which:
  - f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves ($\alpha \geqslant 2$):
  - h_s is a regularization kernel of size s

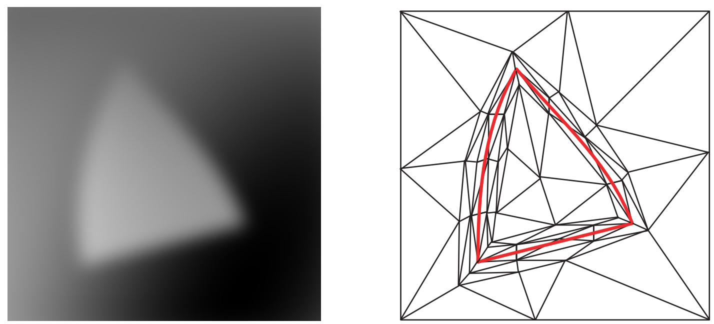


- With M adaptive triangles: $\|f - f_M\|^2 \leqslant C\, M^{-2}$.

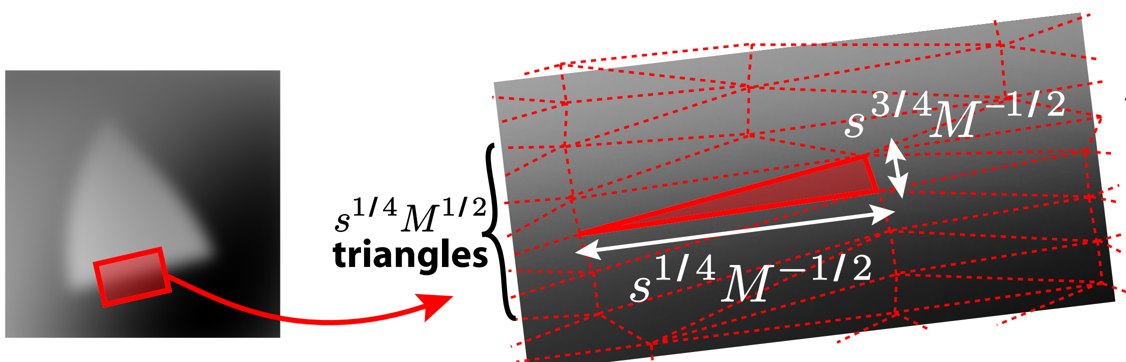


Adaptive Triangulation for Smooth Edges

- Approximations of $f = \tilde{f} \star h_s$ which:
  - f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ “edge” curves ($\alpha \geqslant 2$):
  - h_s is a regularization kernel of size s



- With M adaptive triangles: $\|f - f_M\|^2 \leqslant C\, M^{-2}$.

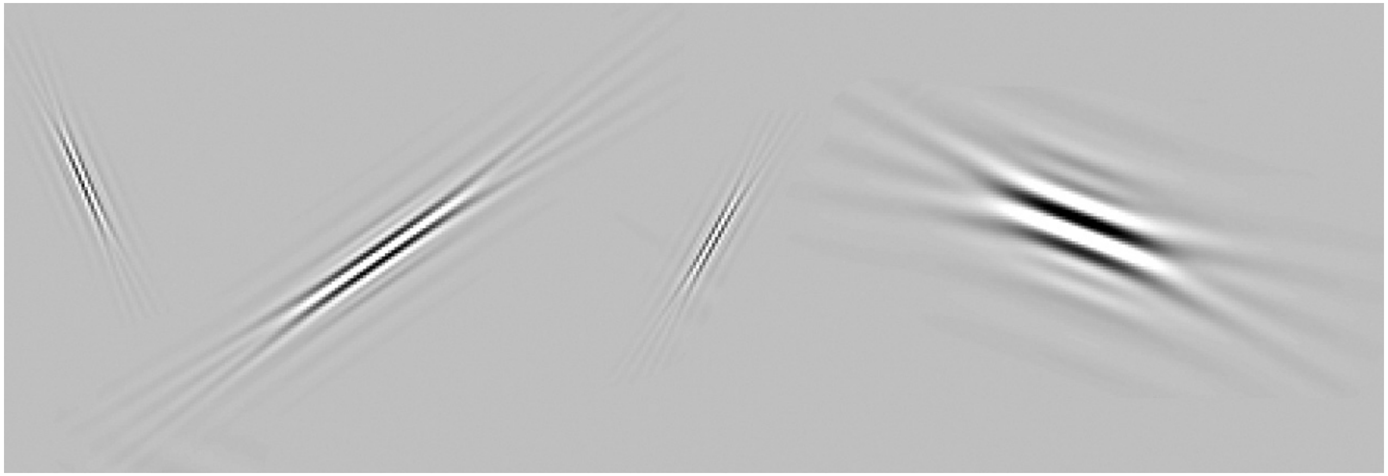


- Difficult to find optimal approximations but good greedy solutions (*Demaret, Dyn, Iske*)

Curvelet Approximation with Edges

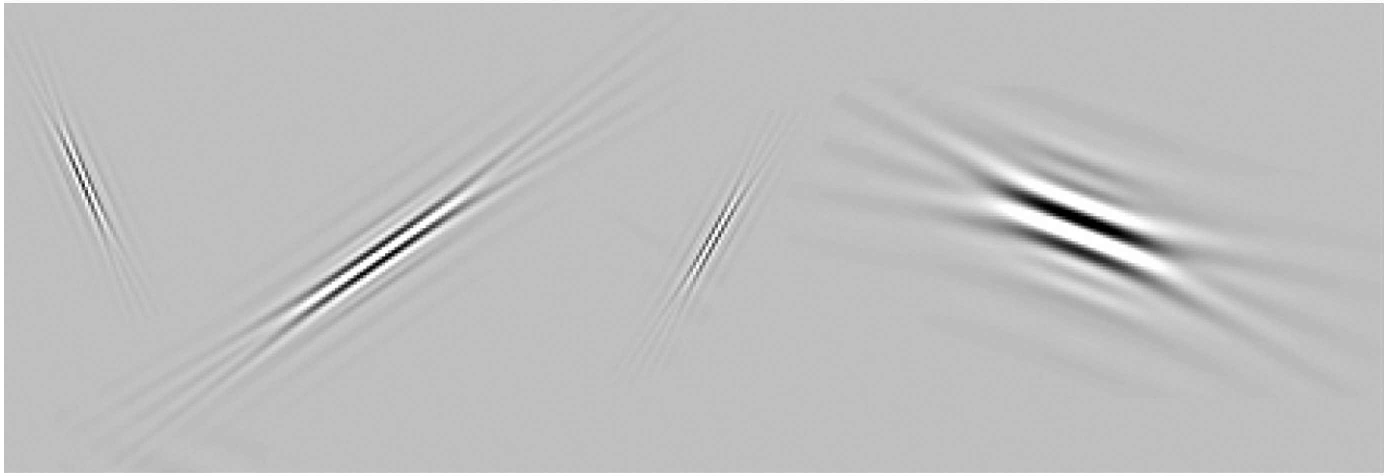
Curvelet Approximation with Edges

- Curvelets define tight frames of $L^2[0,1]^2$ with elongated and rotated elements (*Candes, Donoho*): $\{c_j(R_\theta x - \eta)\}_{j,\theta,\eta}$



Curvelet Approximation with Edges

- Curvelets define tight frames of $L^2[0,1]^2$ with elongated and rotated elements (*Candes, Donoho*): $\{c_j(R_\theta x - \eta)\}_{j,\theta,\eta}$

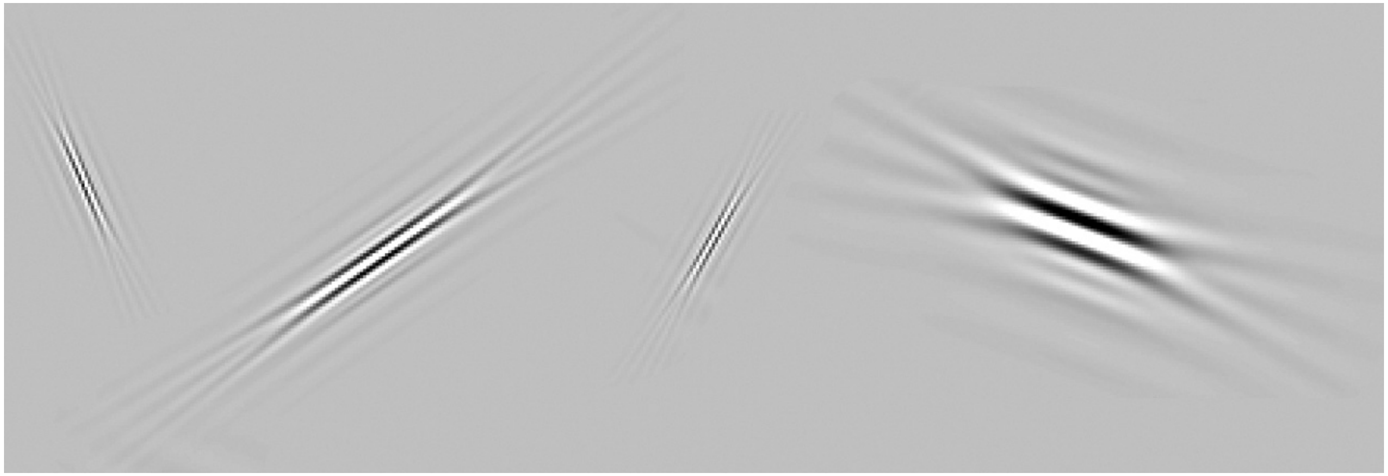


- If f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edges" then with M curvelets:

  if $\alpha \geqslant 2$ then $\|f - f_M\|^2 \leqslant C\,(\log M)^3\, M^{-2}$.

Curvelet Approximation with Edges

- Curvelets define tight frames of $L^2[0,1]^2$ with elongated and rotated elements (*Candes, Donoho*): $\{c_j(R_\theta x - \eta)\}_{j,\theta,\eta}$



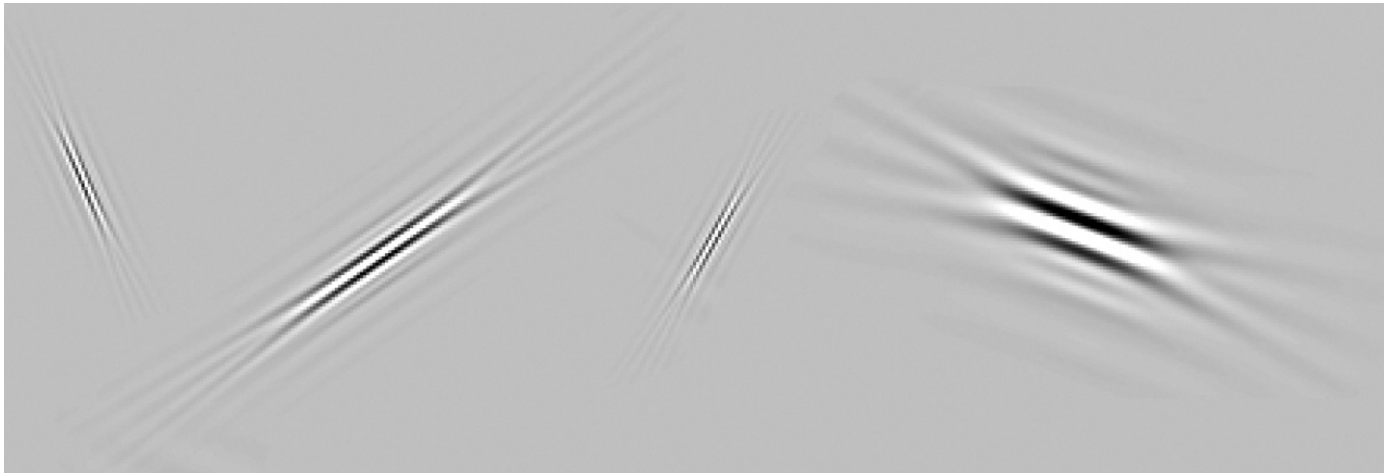
- If f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edges" then with M curvelets:

$$\text{if } \alpha \geqslant 2 \text{ then } \|f - f_M\|^2 \leqslant C\,(\log M)^3\, M^{-2}.$$

- Optimal for $\alpha = 2$.

Curvelet Approximation with Edges

- Curvelets define tight frames of $L^2[0,1]^2$ with elongated and rotated elements (*Candes, Donoho*): $\{c_j(R_\theta x - \eta)\}_{j,\theta,\eta}$



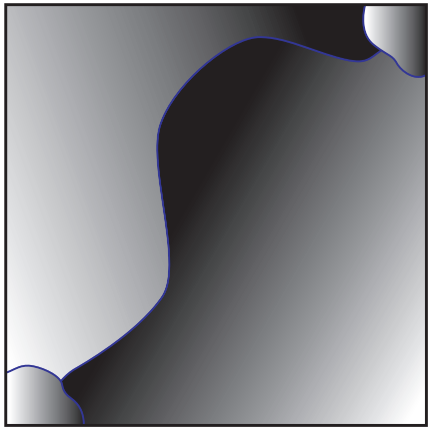
- If f is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edges" then with M curvelets:

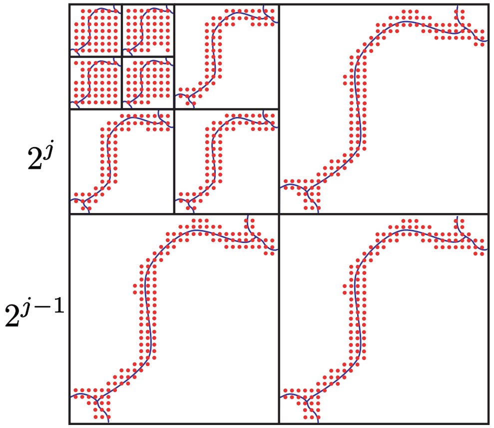
$$\text{if } \alpha \geqslant 2 \text{ then } \|f - f_M\|^2 \leqslant C\,(\log M)^3\, M^{-2}.$$

- Optimal for $\alpha = 2$.
- Difficulty to build discrete orthogonal/Riesz bases: (*Vetterli & Minh Do*).

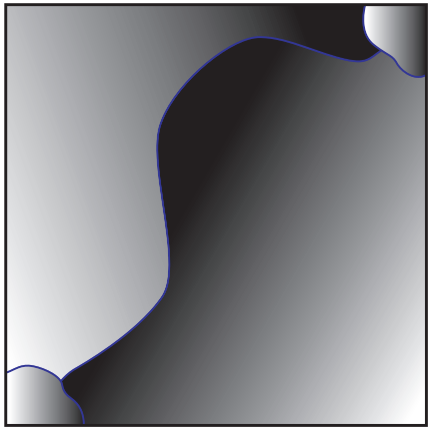
Return to Wavelet Coefficients

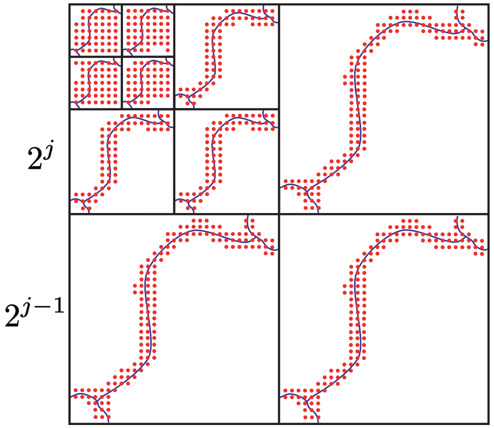
Return to Wavelet Coefficients





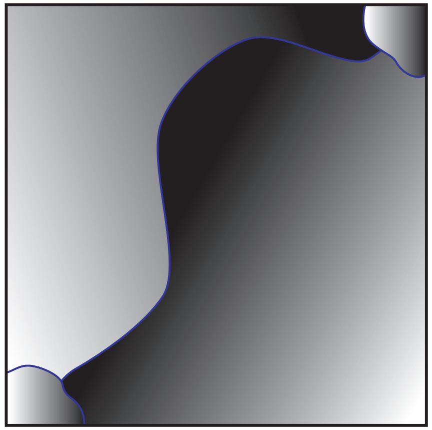
Return to Wavelet Coefficients

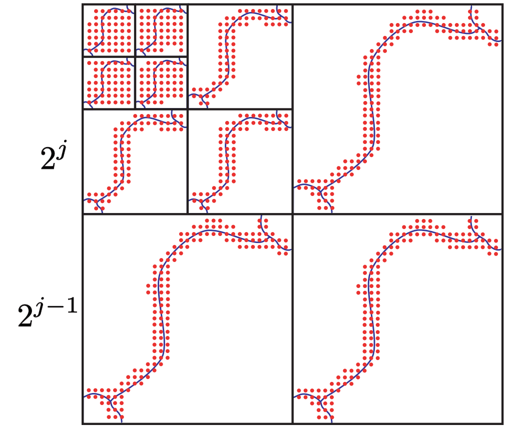




- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?

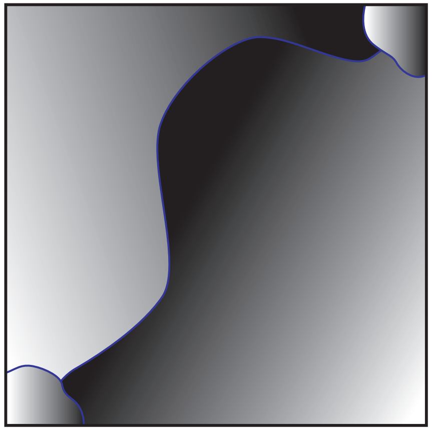
Return to Wavelet Coefficients

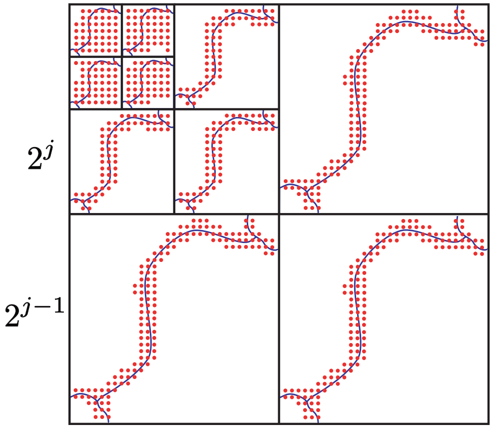




- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.

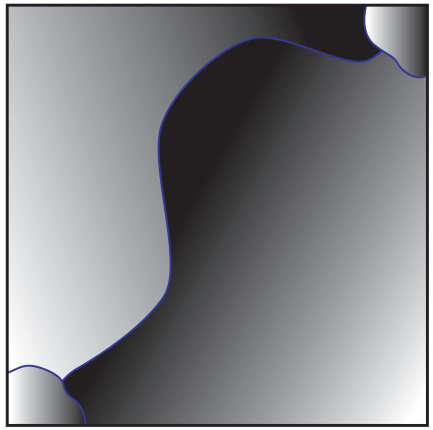
Return to Wavelet Coefficients

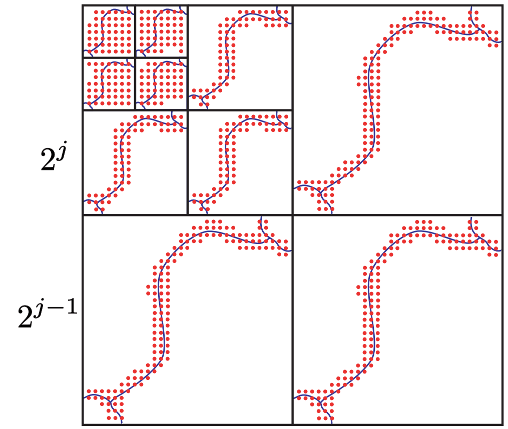




- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: “wedgelets” and “wedgeprints” by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.

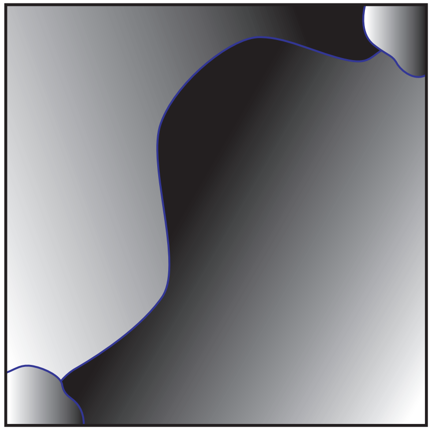
Return to Wavelet Coefficients

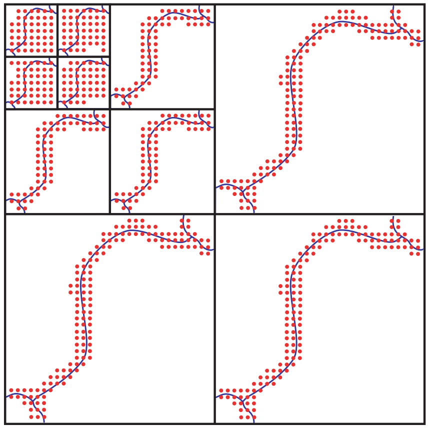




- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.
- Modification of the wavelet transform (*Cohen*).

Return to Wavelet Coefficients



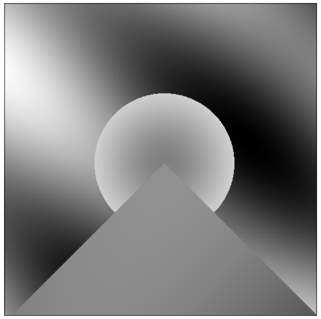


- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.
- Modification of the wavelet transform (*Cohen*).
- Bandelets NG (*Peyré*).

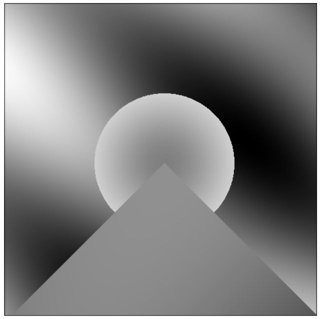
Geometric Model 1

Geometric Model 1

- By parts regular functions with discontinuities along regular curves:

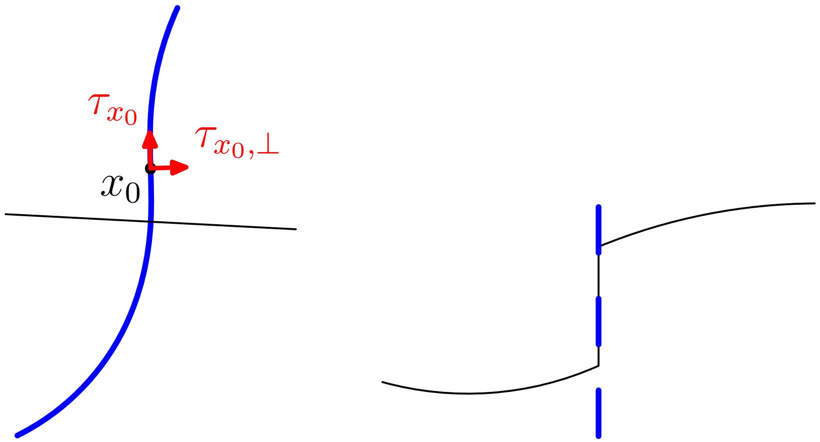


Geometric Model 1



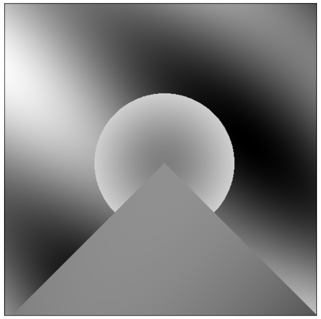
- By parts regular functions with discontinuities along regular curves:

- True discontinuities:

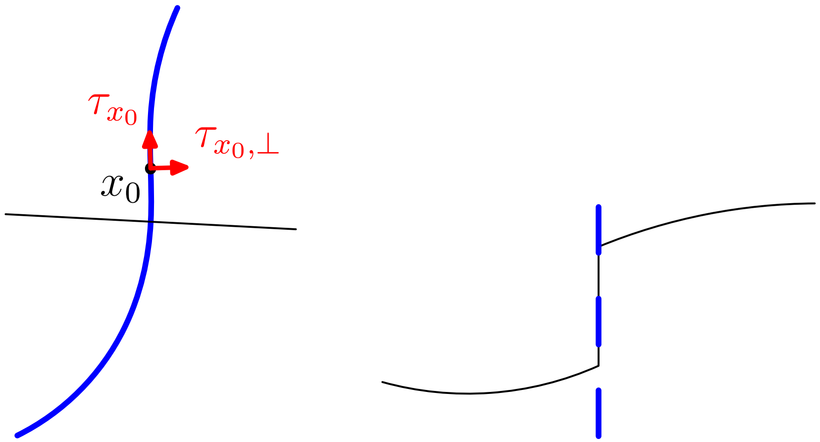


Geometric Model 1

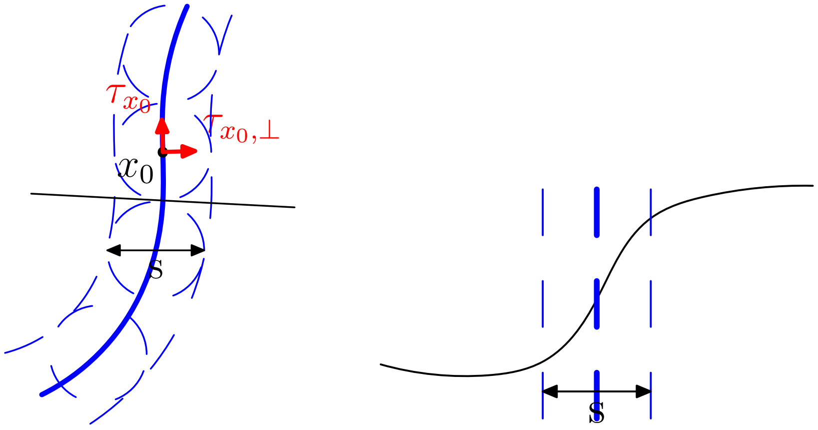
- By parts regular functions with discontinuities along regular curves:



- True discontinuities:

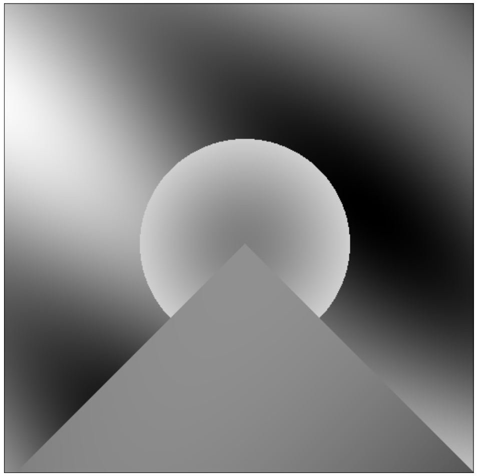


- Smoothed discontinuities:

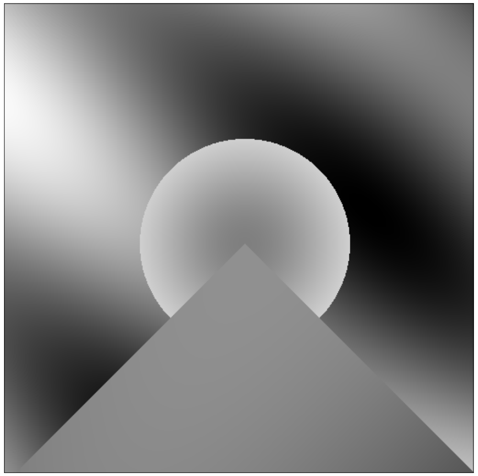


Geometric Model 2

Geometric Model 2

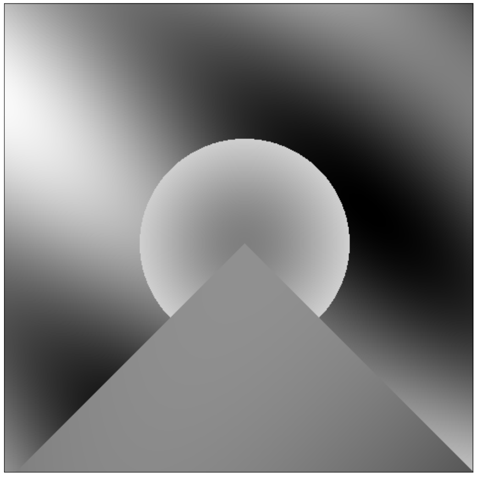


Geometric Model 2



- $\mathbf{C}^{\alpha}$ Horizon Model of Donoho revisited.

Geometric Model 2



- $\mathbf{C}^\alpha$ Horizon Model of Donoho revisited.
- $\mathbf{C}^\alpha$ Geometrically Regular:
  - $f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f} \in \mathbf{C}^\alpha(\Lambda)$ for $\Lambda = [0,1]^2 - \{\mathcal{C}_\gamma\}_{1 \leqslant \gamma \leqslant G}$,
  - the blurring kernel h is $\mathbf{C}^\alpha$, compactly supported in $[-s,s]^2$ and $\|h\|_{\mathbf{C}^\alpha} \leqslant s^{-(2+\alpha)}$.
  - the edge curves $\mathcal{C}_\gamma$ are α differentiable and do not intersect tangentially.

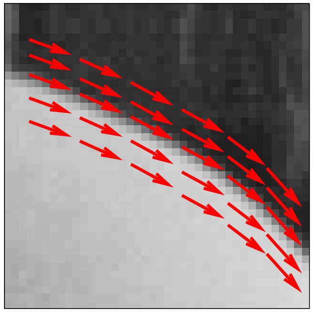
Geometric Flow

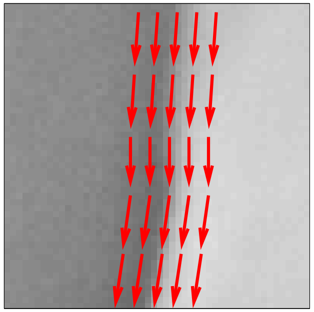
Geometric Flow

- Geometric flow: vector field $\vec{\tau}(x_1, x_2)$ giving local direction of regularity of the image.

Geometric Flow

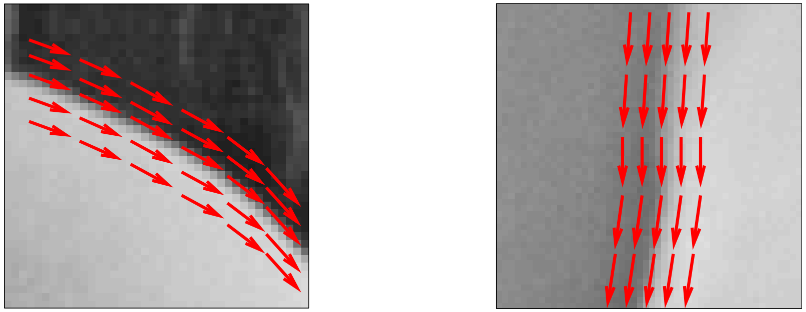
- Geometric flow: vector field $\vec{\tau}(x_1, x_2)$ giving local direction of regularity of the image.
- In a region, the flow is either vertically or horizontally parallel.



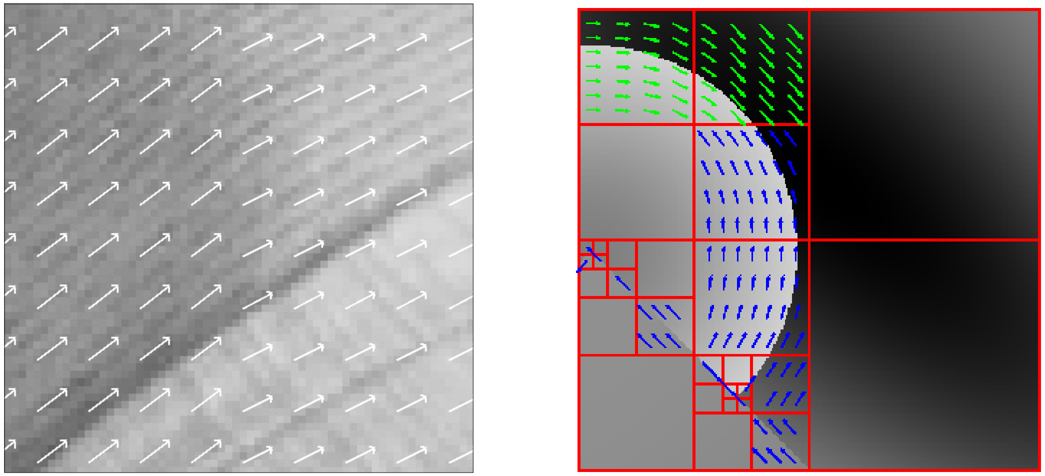


Geometric Flow

- Geometric flow: vector field $\vec{\tau}(x_1, x_2)$ giving local direction of regularity of the image.
- In a region, the flow is either vertically or horizontally parallel.



- The image is segmented in such regions:

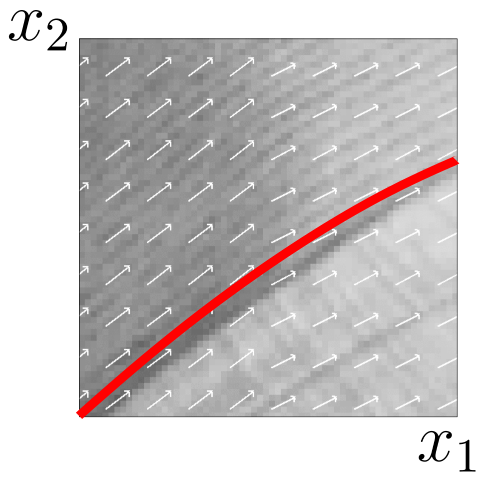


Warped Wavelet Basis

Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$

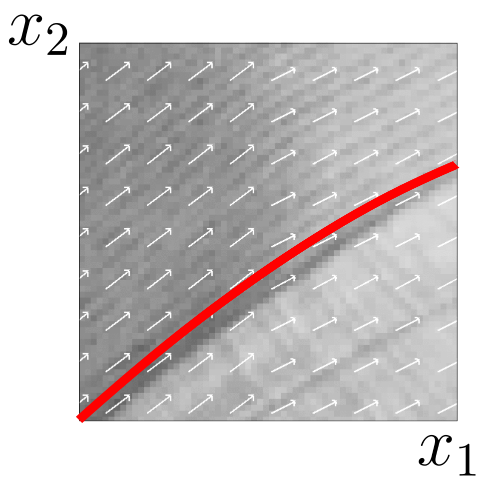


$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

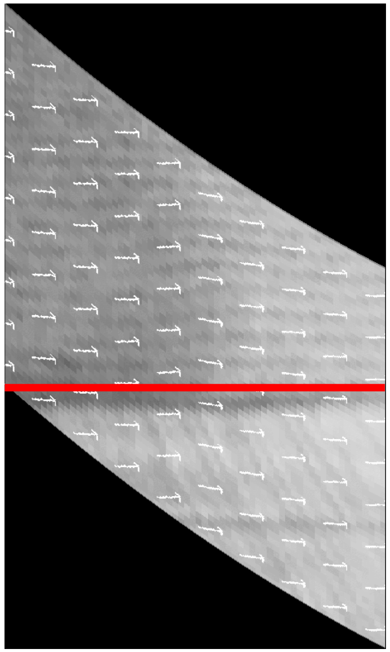
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

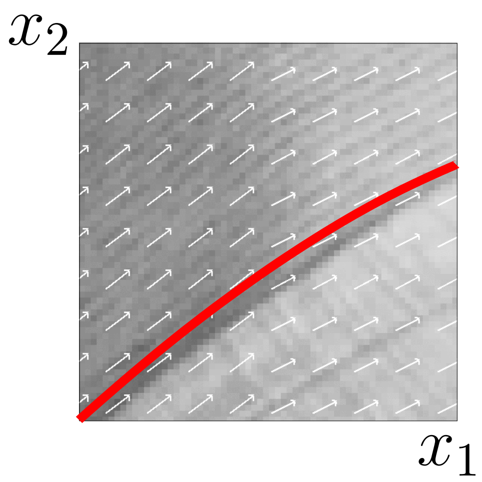


- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.

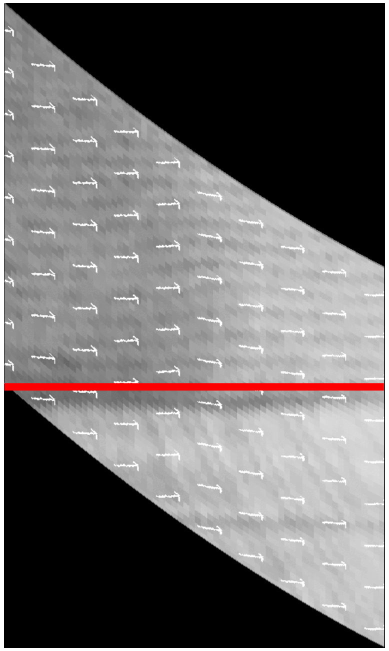
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\, \mathrm{d}u$$

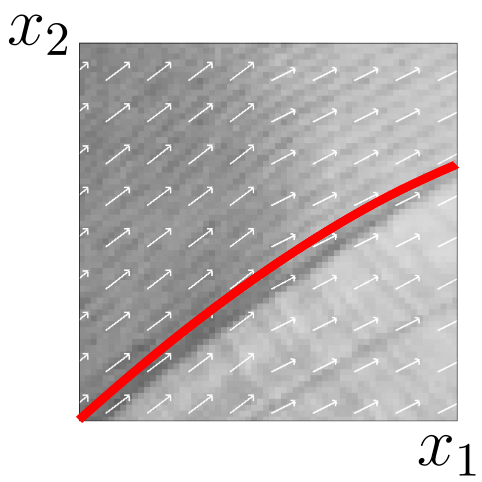


- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.
- $\langle f(x_1, x_2 + c(x_1))\,,\, \Psi(x_1, x_2)\rangle = \langle f(x_1, x_2)\,,\, \Psi(x_1, x_2 - c(x_1))\rangle$

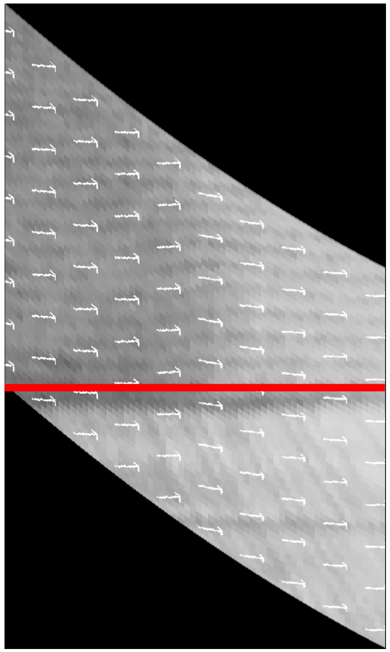
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\,c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$



- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.
- $\langle f(x_1, x_2 + c(x_1))\,,\,\Psi(x_1, x_2)\rangle = \langle f(x_1, x_2)\,,\,\Psi(x_1, x_2 - c(x_1))\rangle$
- Decomposition in a *warped wavelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ll} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & ,\quad \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & ,\quad \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}.$$

Bandeletization

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).
- Bandeletization: replace $\{\phi_{j,m_1}(x_1)\}_{m_1}$ with a wavelet family $\{\psi_{l,m_1}(x_1)\}_{l>j\,,m_1}$ that spans the same space.
- Warped wavelet basis of $L^2(\Omega)$:

$$\left\{ \begin{array}{lcl} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2-c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2-c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2-c(x_1)) \end{array} \right\}_j$$

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).
- Bandeletization: replace $\{\phi_{j,m_1}(x_1)\}_{m_1}$ with a wavelet family $\{\psi_{l,m_1}(x_1)\}_{l>j\,,m_1}$ that spans the same space.
- *Bandelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \psi_{l,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}_{\substack{j,l>j \\ m_1,m_2}} .$$

Anisotropic

d

Segmented Bandelet Basis

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

- Fast bandelet transform ($O(N^2)$):
  - resampling, fast warped wavelet transform, bandeletization.

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

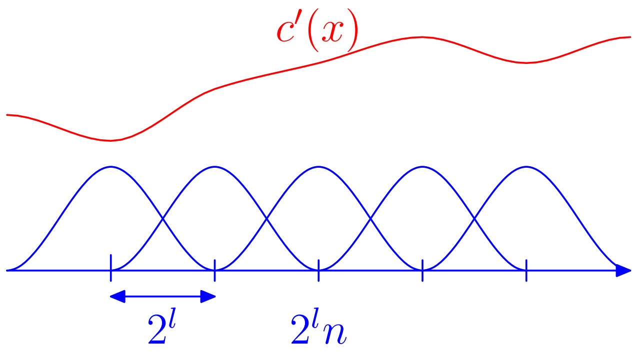
- Fast bandelet transform $(O(N^2))$:
  - resampling, fast warped wavelet transform, bandeletization.
- No blocking effect with an adapted lifting scheme.

Flow Determination

Flow Determination

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$

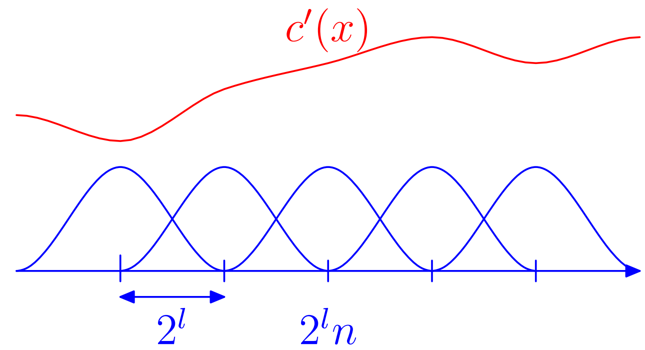


and the $L\,2^{-l}$ parameters α_n.

Flow Determination

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

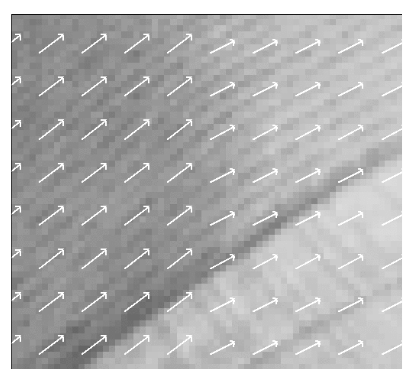
$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$



and the $L\,2^{-l}$ parameters α_n.

- Minimization of

$$\int_\Omega \left| \nabla f(x_1, x_2) \, . \, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \; .$$



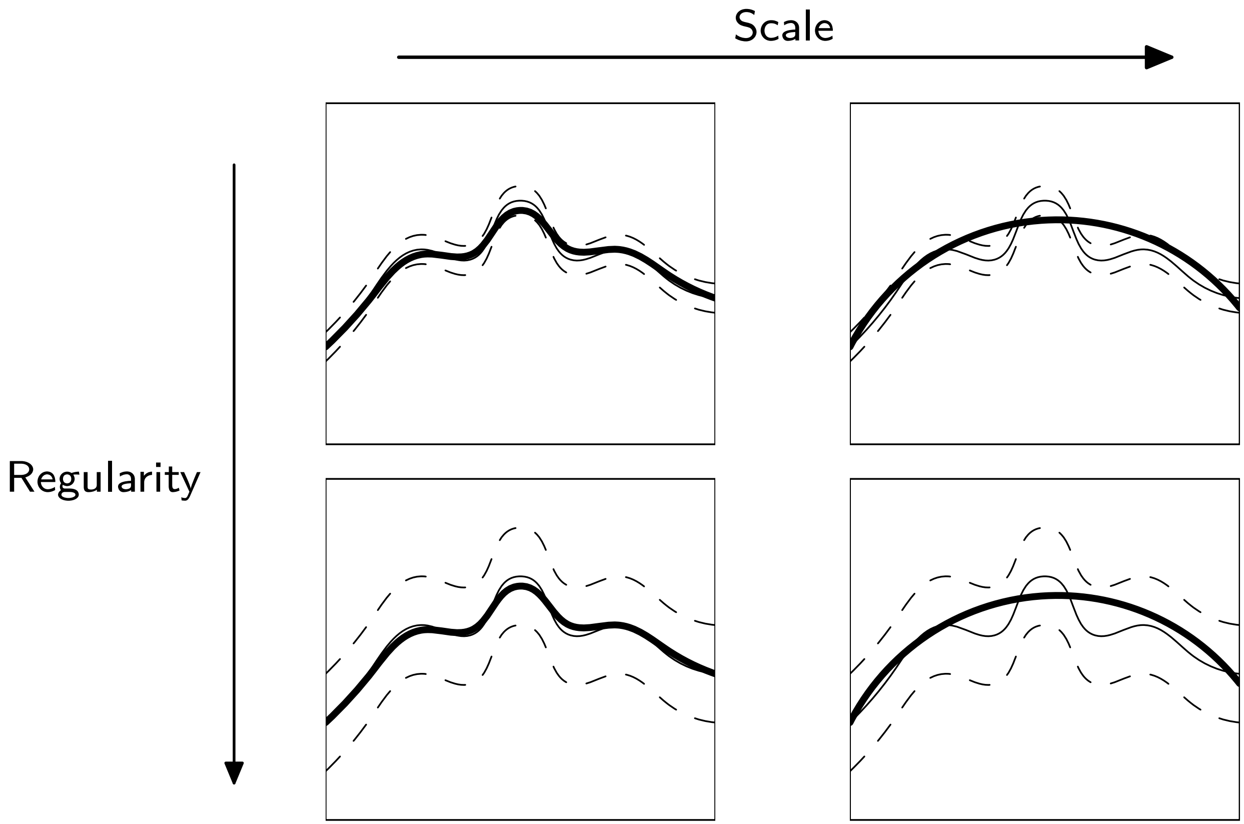
Choice of Parameterization Scale

Choice of Parameterization Scale

- Scale 2^l adapted to the regularity of the image along the flow:

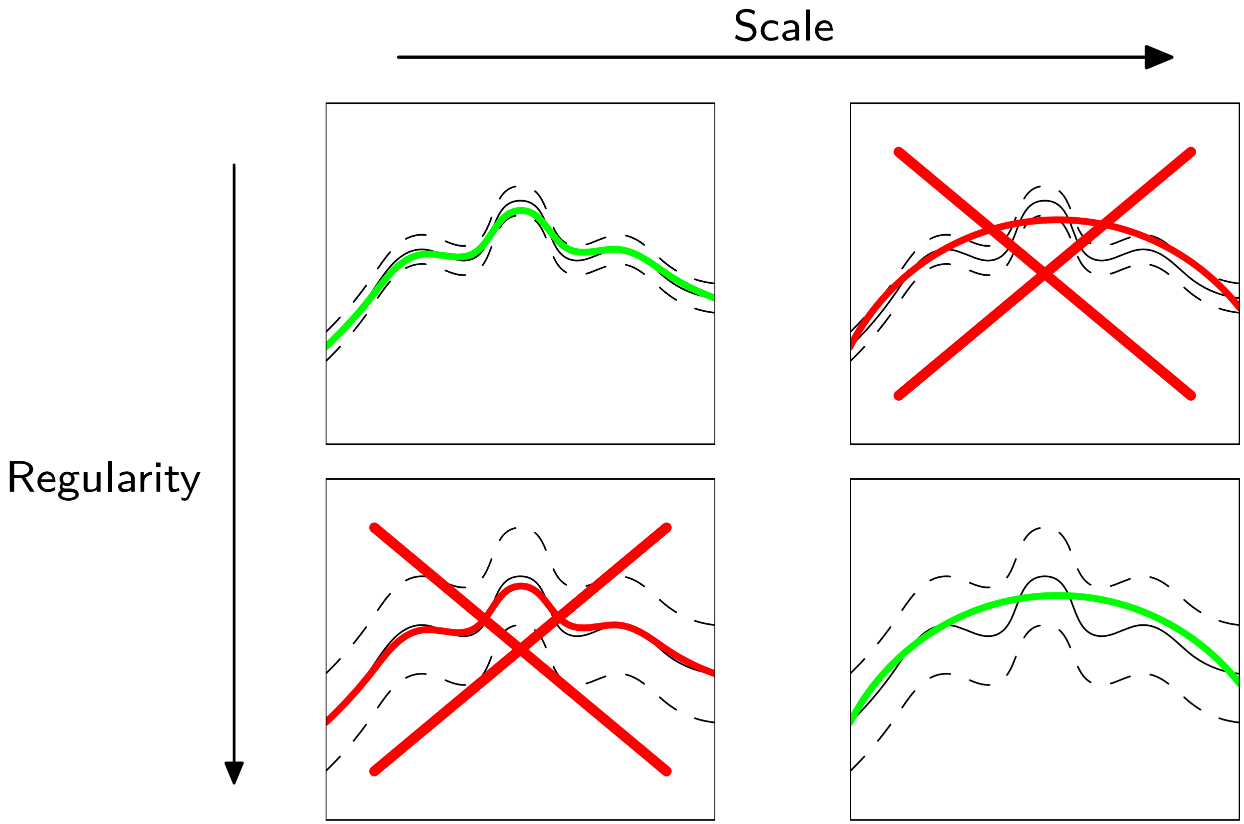
Choice of Parameterization Scale

- Scale 2^l adapted to the regularity of the image along the flow:



Choice of Parameterization Scale

- Scale 2^l adapted to the regularity of the image along the flow:



M Term Approximation

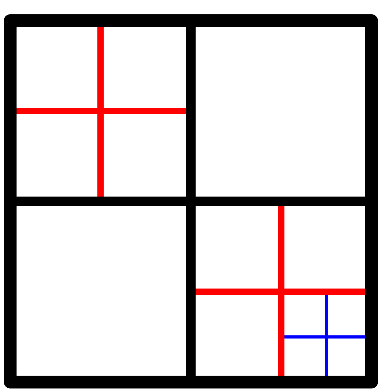
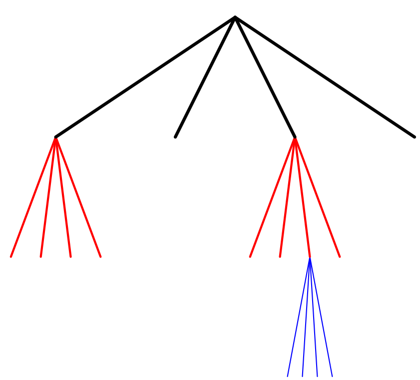
M Term Approximation

- A bandelet approximation is specified by:
  - a dyadic squares segmentation given by the M_s interior nodes of a quadtree,
  - and inside each square Ω_i of the segmentation by::
    - $M_{g,i}$ coefficients for the geometric flow,
    - $M_{b,i}$ bandelets coefficients above a threshold T.

M Term Approximation

- A bandelet approximation is specified by:
  - a dyadic squares segmentation given by the M_s interior nodes of a quadtree,
  - and inside each square Ω_i of the segmentation by::
    - $M_{g,i}$ coefficients for the geometric flow,
    - $M_{b,i}$ bandelets coefficients above a threshold T.
- Total number of parameters:

$$M = M_s + \sum_i \left(M_{g,i} + M_{b,i} \right) .$$

Optimization

Optimization

- Minimization of $\|f - f_M\|^2$ for a given number of parameters M.

Optimization

- Minimization of $\|f - f_M\|^2$ for a given number of parameters M.
- Lagrangian approach: best geometric segmented flow that minimizes

$$\|f - f_M\|^2 + T^2 \, M \ .$$

Optimization

- Minimization of $\|f - f_M\|^2$ for a given number of parameters M.
- Lagrangian approach: best geometric segmented flow that minimizes

$$\|f - f_M\|^2 + T^2 M \ .$$

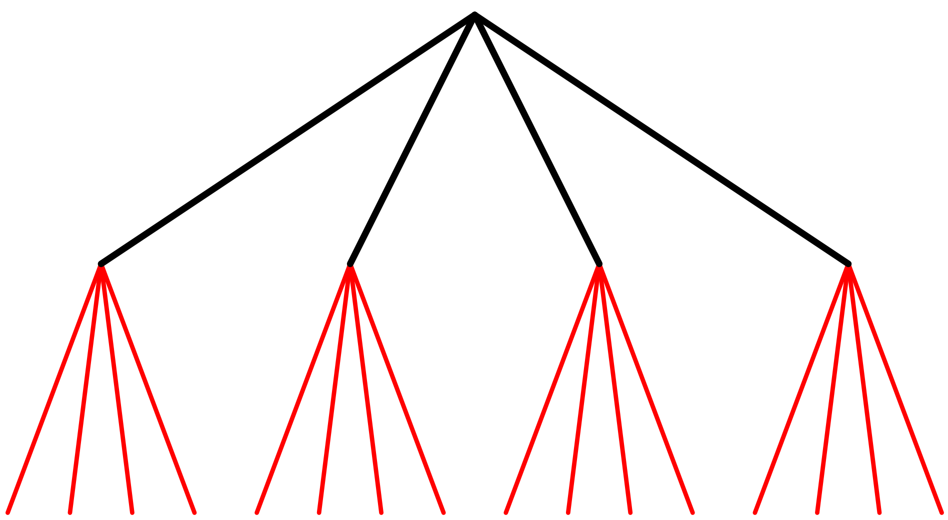
- Fast algorithm (CART): Bottom to top dynamic programming on the quadtree segmentation.

Optimization

- Minimization of $\|f - f_M\|^2$ for a given number of parameters M.
- Lagrangian approach: best geometric segmented flow that minimizes

$$\|f - f_M\|^2 + T^2\, M \ .$$

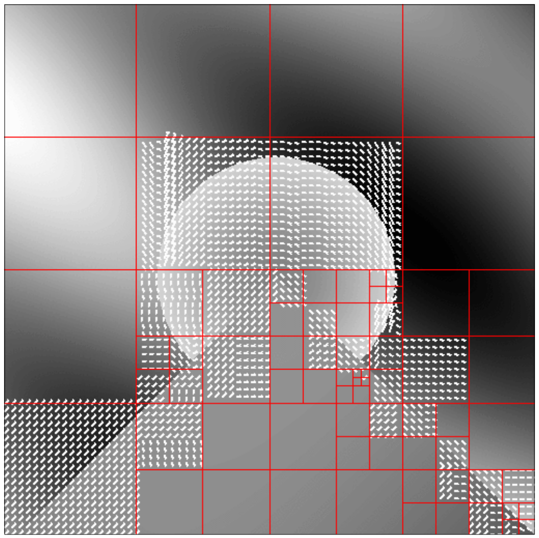
- Fast algorithm (CART): Bottom to top dynamic programming on the quadtree segmentation.
- Complexity: $O(N^2\,(\log N)^2)$ for N^2 pixels.



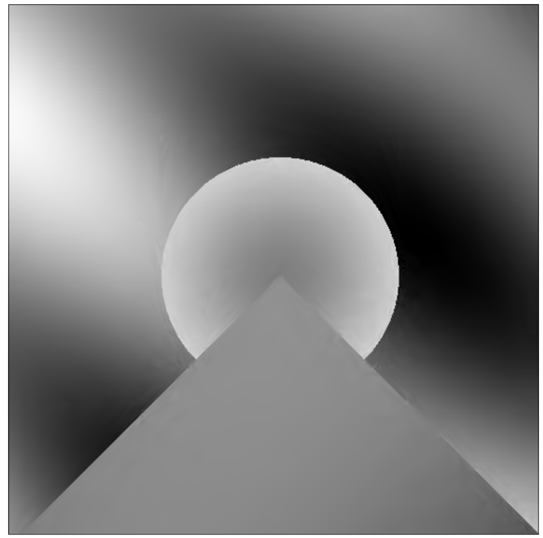
Results

Results

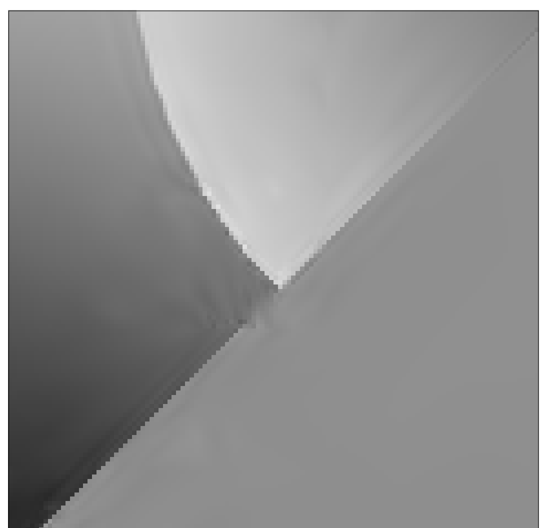
M=2650



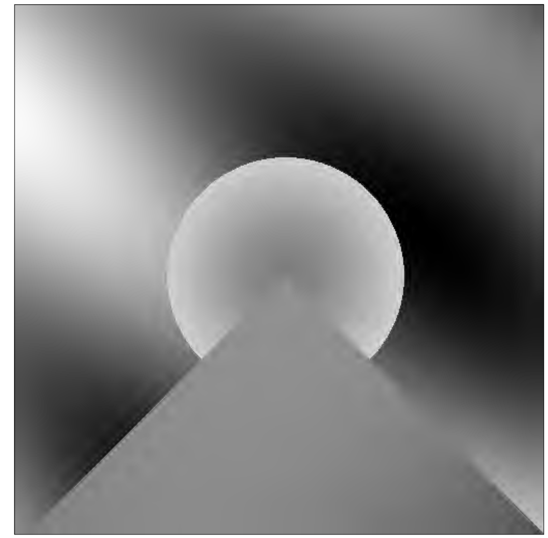
PSNR = $45{,}97\,\mathrm{dB}$



Bandelets



PSNR = $40{,}17\,\mathrm{dB}$



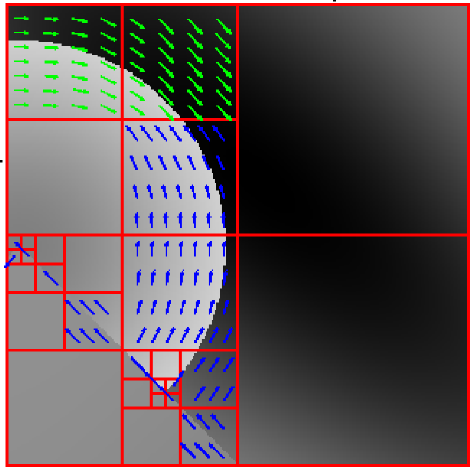
Wavelets

Theorem

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions) then

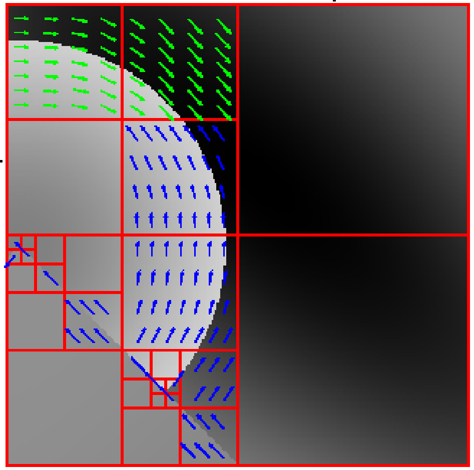
$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} .$$



Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions) then

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$

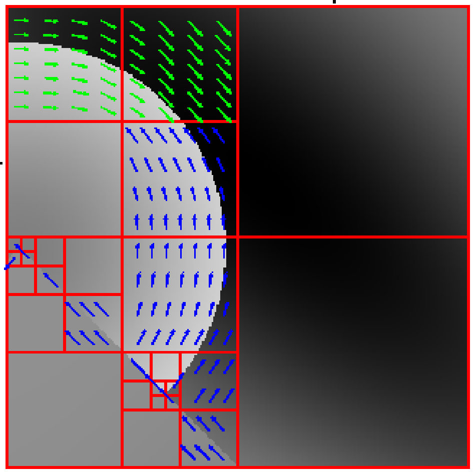


- Unknown degree of smoothness α.

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions) then

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$

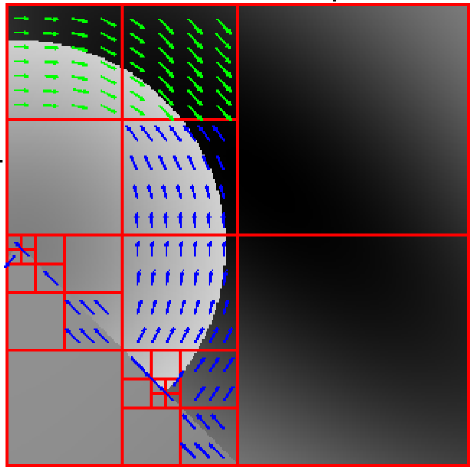


- Unknown degree of smoothness α.
- Optimal decay exponent α.

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions) then

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \;.$$



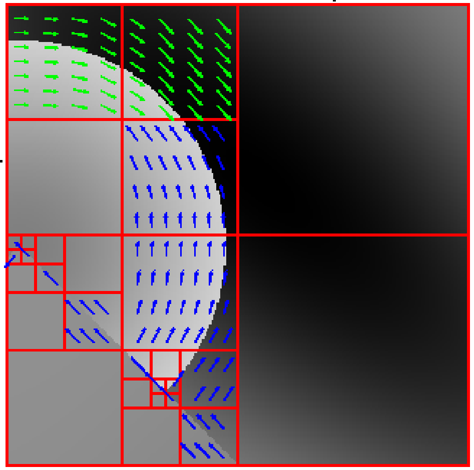
- Unknown degree of smoothness α.
- Optimal decay exponent α.
- Improvement over isotropic wavelets for which

$$\|f - f_M\|^2 \leqslant C\, M^{-1} \quad .$$

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions) then

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



- Unknown degree of smoothness α.
- Optimal decay exponent α.
- Improvement over isotropic wavelets for which

$$\|f - f_M\|^2 \leqslant C\, M^{-1} \quad .$$

- Improvement over curvelets for which

$$\|f - f_M\|^2 \leqslant C\, (\log M)^3\, M^{-2} \quad .$$

Image Compression

Image Compression

- A compressed image $\tilde{f}$ is calculated from f by:

Image Compression

- A compressed image $\tilde{f}$ is calculated from f by:
  - Quantizing uniformly all bandelet coefficients with step size Δ.

Image Compression

- A compressed image $\tilde{f}$ is calculated from f by:
  - Quantizing uniformly all bandelet coefficients with step size Δ.
  - Entropy coding all parameters with a total of R bits.

Image Compression

- A compressed image $\tilde{f}$ is calculated from f by:
  - Quantizing uniformly all bandelet coefficients with step size Δ.
  - Entropy coding all parameters with a total of R bits.
  - Optimizing the geometric flow to minimize the Lagrangian:

$$\|f - \tilde{f}\|^2 \; + \lambda \, \Delta^2 \, R \quad \text{with} \quad \lambda \approx 0.107$$

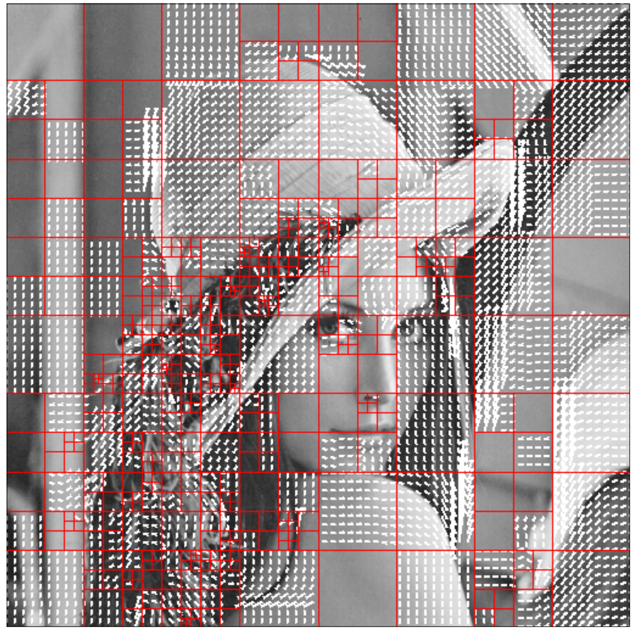
Image Compression

- A compressed image $\tilde{f}$ is calculated from f by:
  - Quantizing uniformly all bandelet coefficients with step size Δ.
  - Entropy coding all parameters with a total of R bits.
  - Optimizing the geometric flow to minimize the Lagrangian:

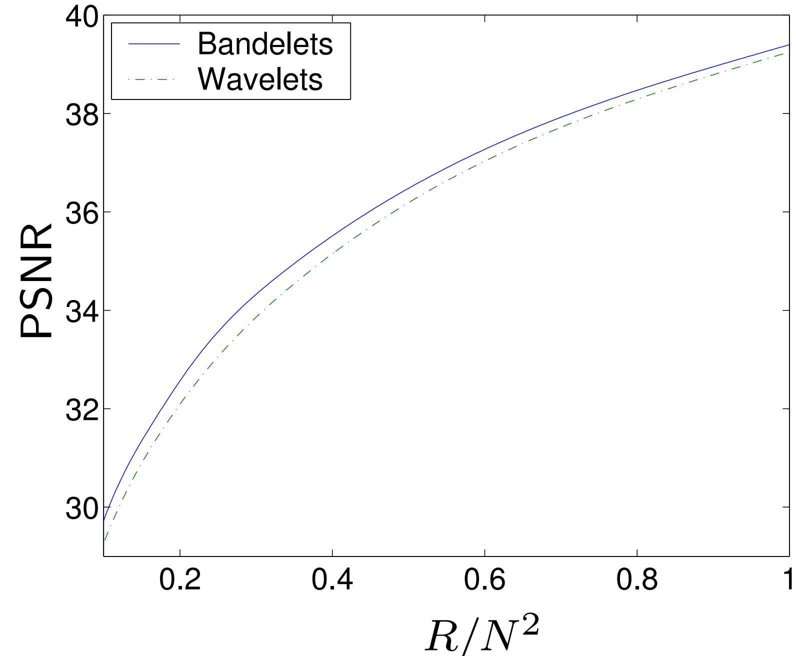
$$\|f - \tilde{f}\|^2 + \lambda \, \Delta^2 \, R \quad \text{with} \quad \lambda \approx 0.107$$

- $O(N^2 (\log_2 N)^2)$ operations.

Original



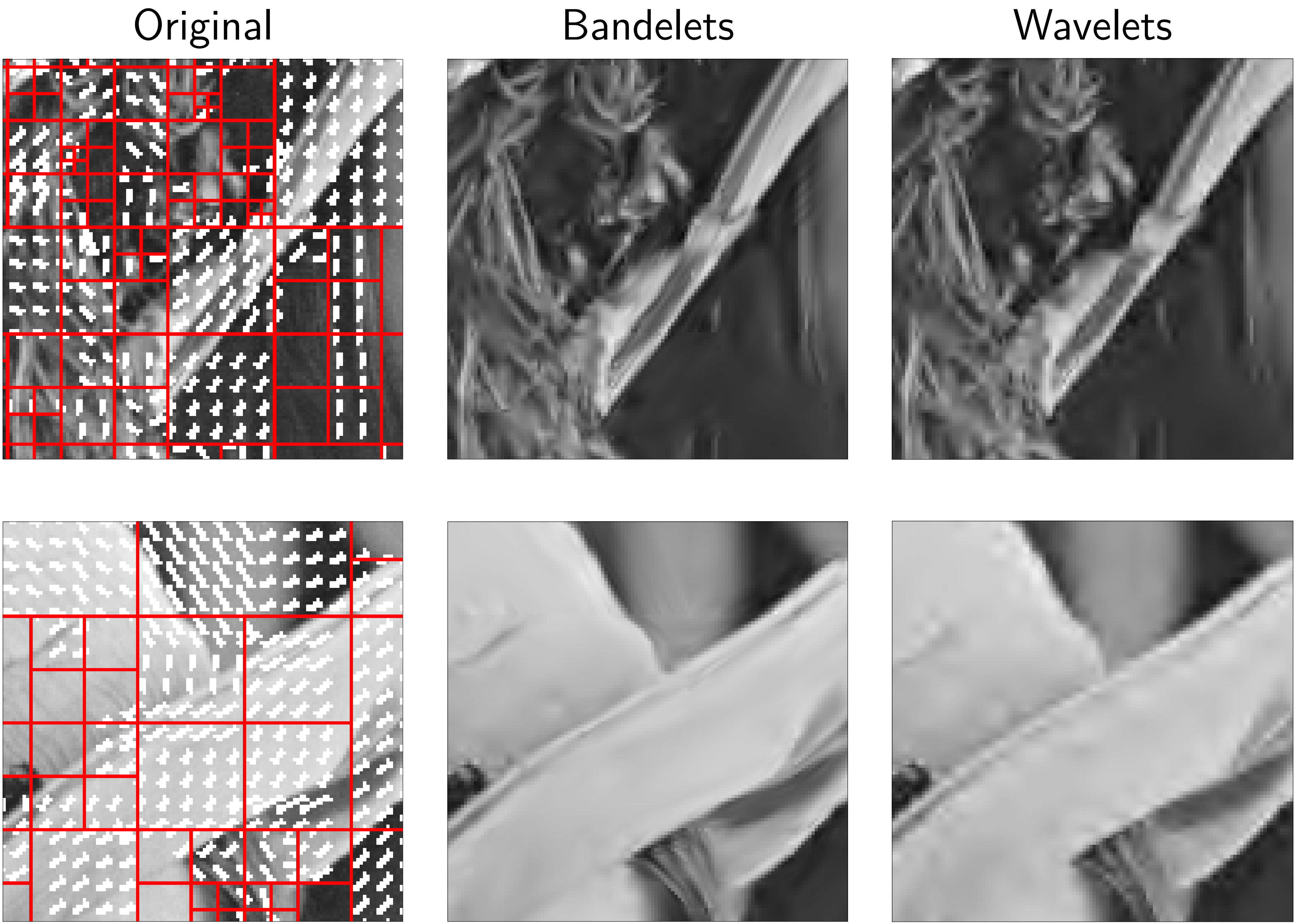
Distortion-Rate



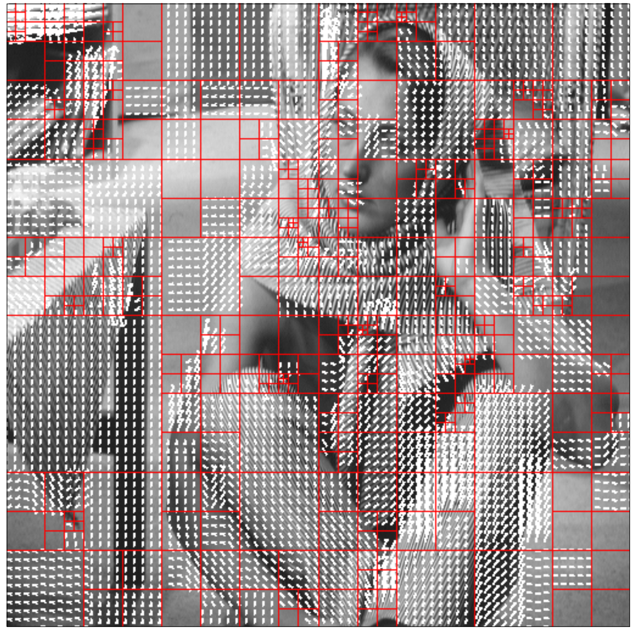
$R/N^2 = 0.22$ bpp

Bandelets (33.05 db)

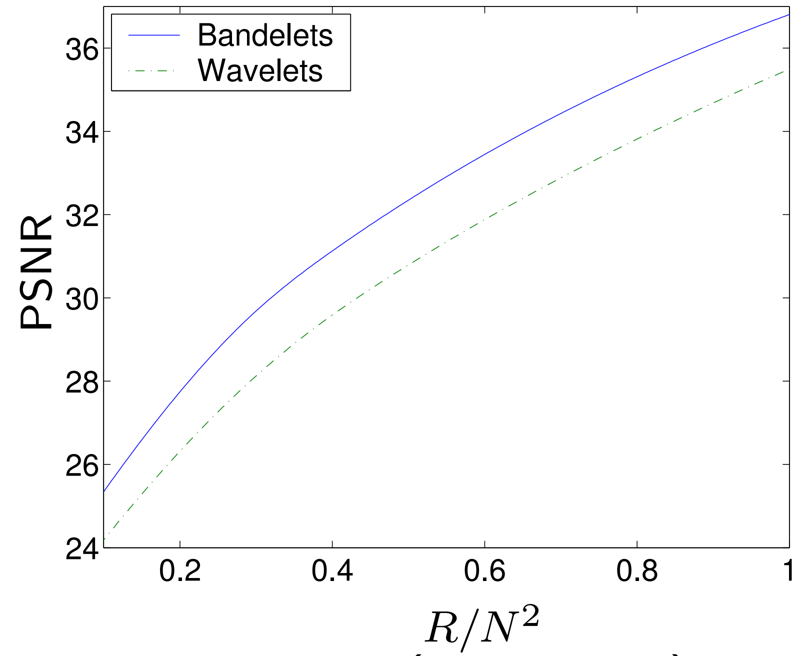
Wavelets (32.54 db)


Original
Bandelets
Wavelets

Original



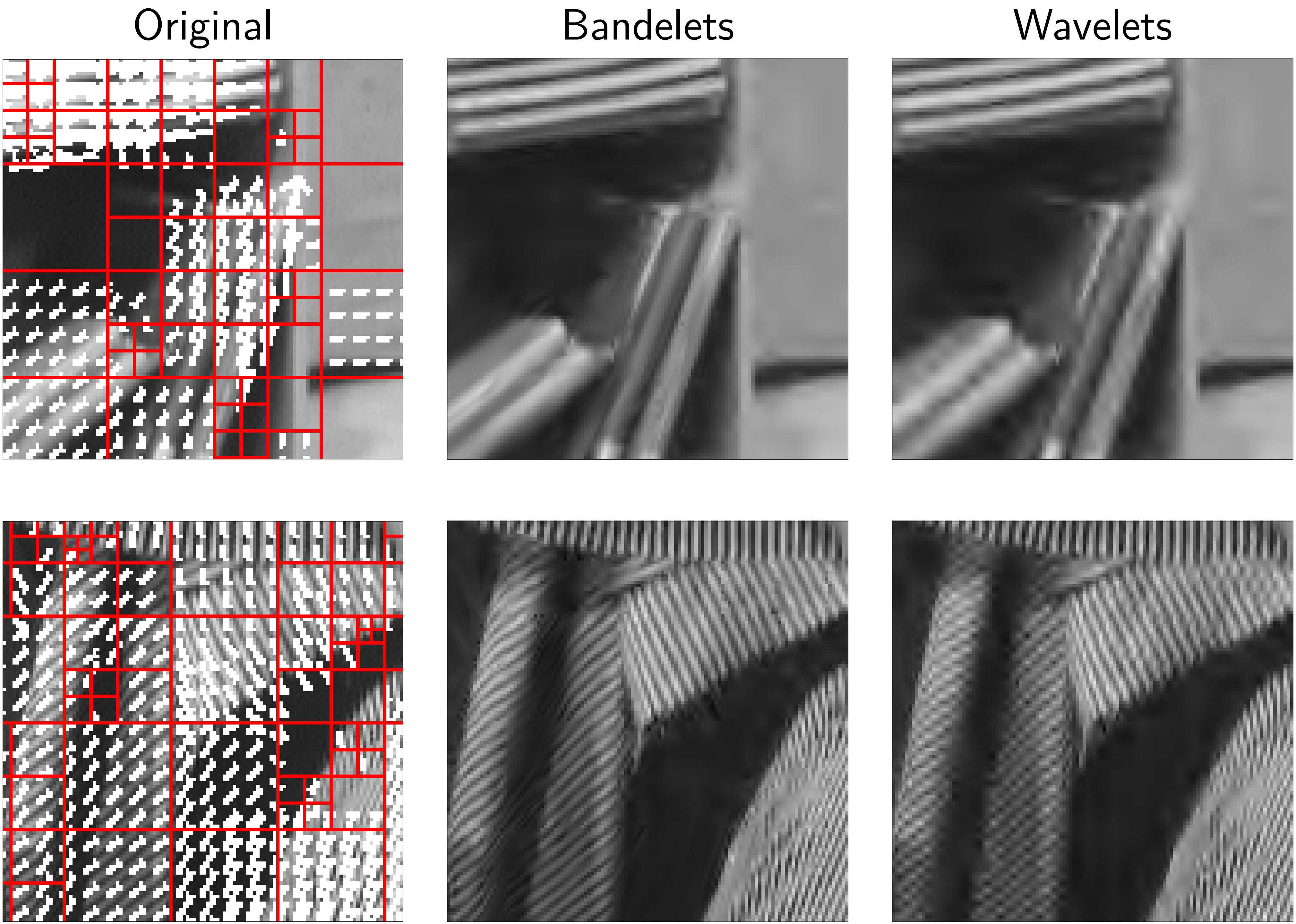
Distortion-Rate



$R/N^2 = 0.40$ bpp

Bandelets (31.22 db)

Wavelets (29.68 db)


Original
Bandelets
Wavelets

Denoising

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.

Denoising

- Estimate an image f from the noisy data

$$X = f + W \quad \text{where } W \text{ is Gaussian white of variance } \sigma^2.$$

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.
- The *oracle model* minimizes the risk $\mathsf{E}\left(\|F - f\|^2\right)$.

Denoising

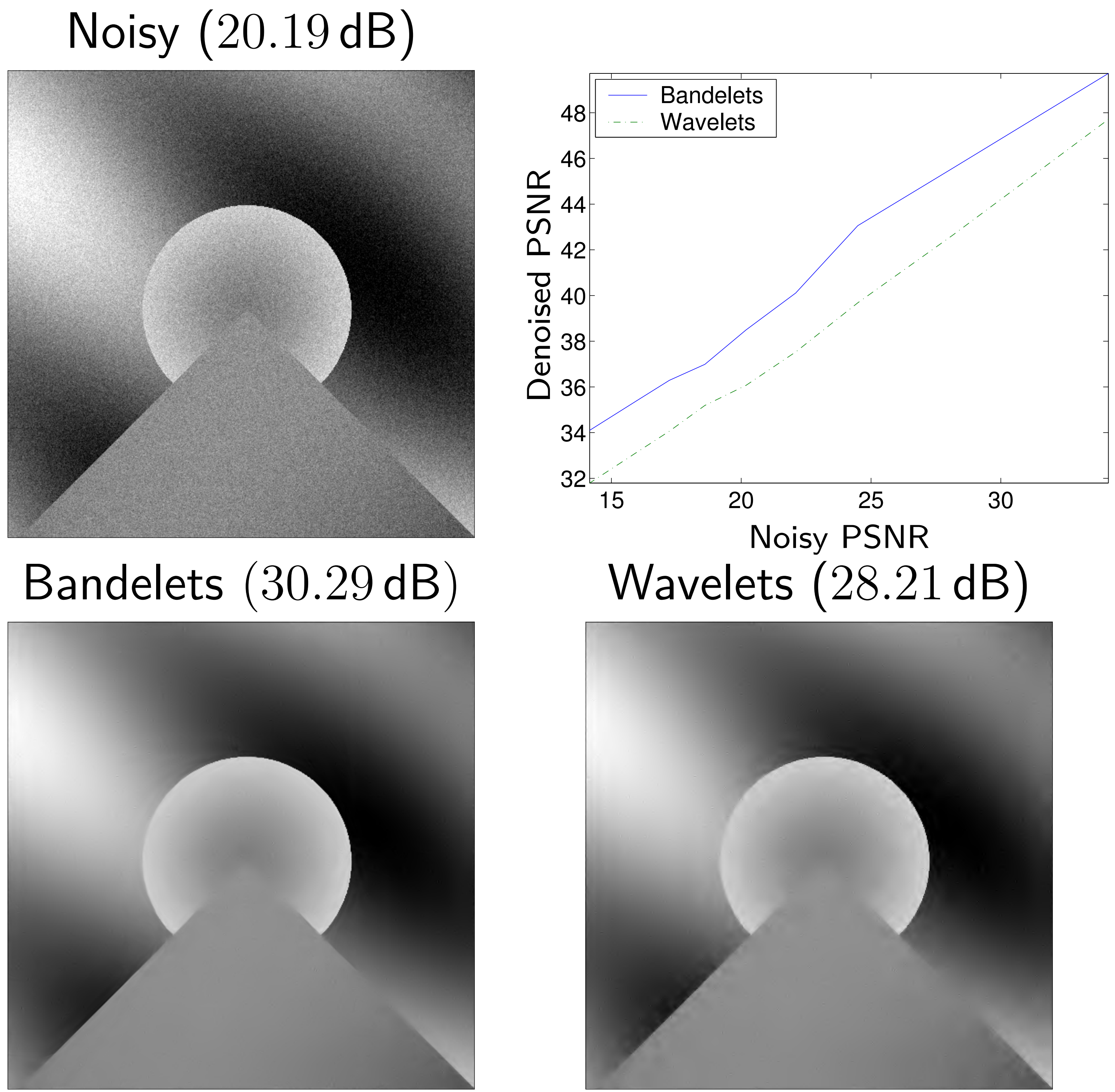
- Estimate an image f from the noisy data

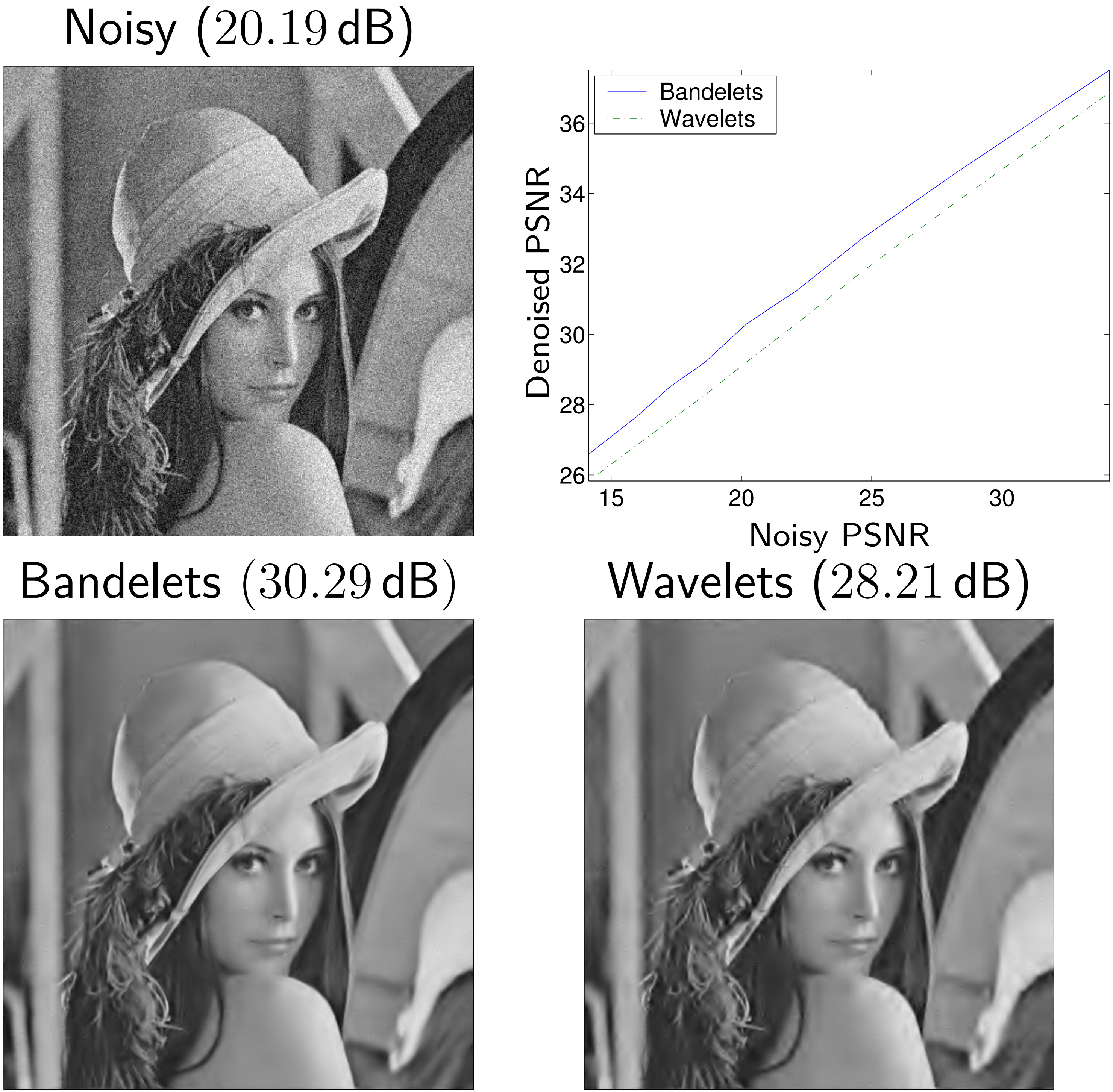
  $X = f + W$ where W is Gaussian white of variance σ^2.

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

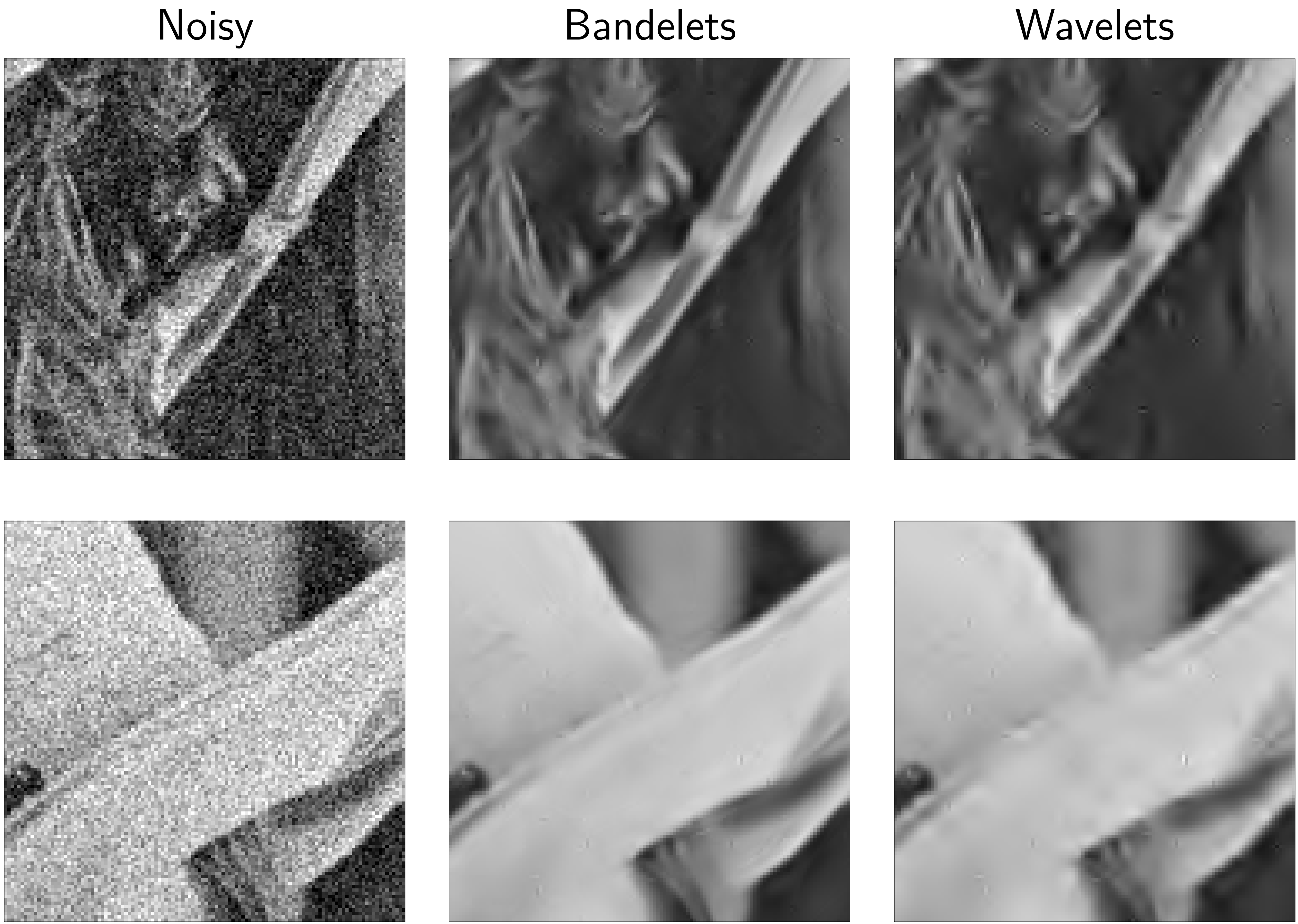
$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \; .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.
- The *oracle model* minimizes the risk $\mathsf{E}\left(\|F - f\|^2\right)$.
- Design of a penalized estimator :
  - MDL : $\|X - F\|^2 + \lambda\sigma^2 R$.
  - Complexity : $-\|F\|^2 + \lambda\sigma^2 M$.

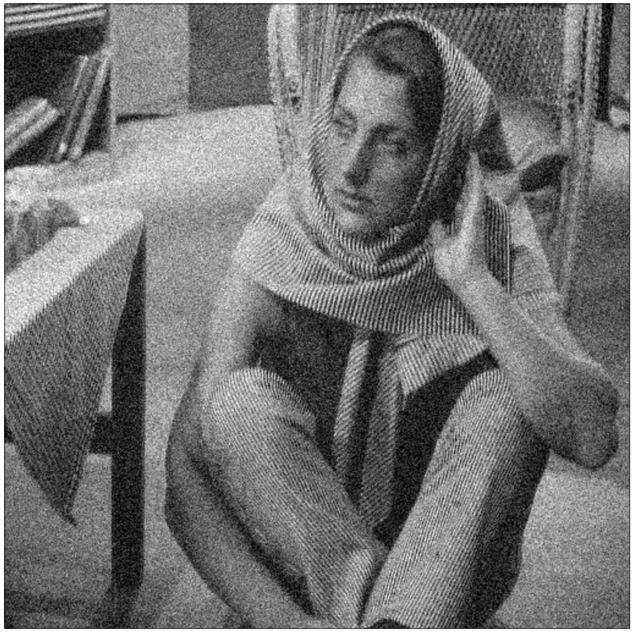

Noisy (20.19 dB)
Bandelets
Wavelets
Denoised PSNR
Noisy PSNR
48
46
44
42
40
38
36
34
32
15
20
25
30
Bandelets (30.29 dB)
Wavelets (28.21 dB)

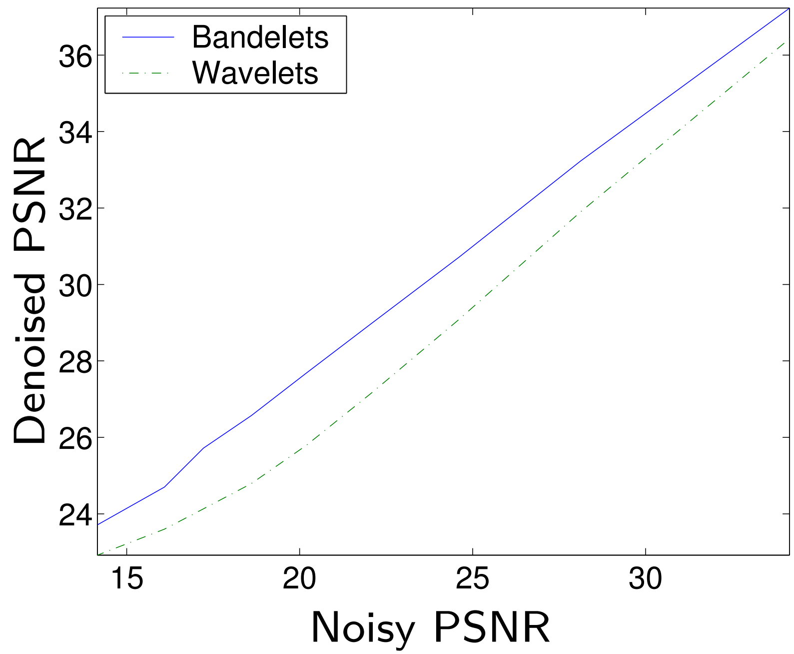


Noisy (20.19 dB)
Bandelets
Wavelets
36
34
32
30
28
26
15
20
25
30
Denoised PSNR
Noisy PSNR
Bandelets (30.29 dB)
Wavelets (28.21 dB)


Noisy
Bandelets
Wavelets

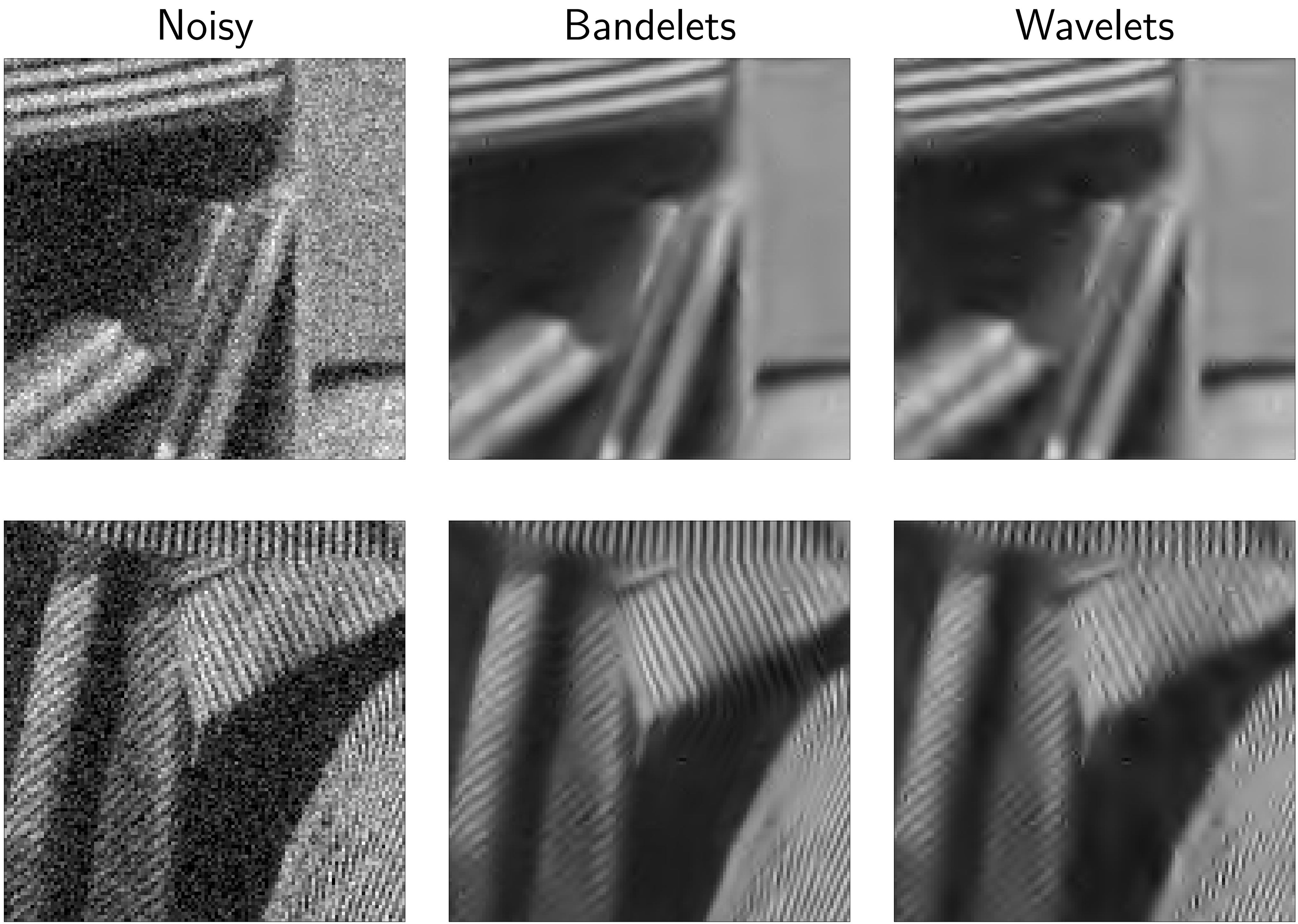
Noisy ($20.19\,\text{dB}$)





Bandelets ($27.68\,\text{dB}$)

Wavelets ($25.79\,\text{dB}$)


Noisy
Bandelets
Wavelets

Conclusion

Conclusion

- Bandelets provide a sparse representation of images in basis adapted to the geometry.

Conclusion

- Bandelets provide a sparse representation of images in basis adapted to the geometry.
- Suitable for applications.

Conclusion

- Bandelets provide a sparse representation of images in basis adapted to the geometry.
- Suitable for applications.
- Theoretical framework for proof.

Conclusion

- Bandelets provide a sparse representation of images in basis adapted to the geometry.
- Suitable for applications.
- Theoretical framework for proof.
- Tomorrow : Implementation and theory.

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. Dossal)
  - Denoising
  - Deconvolution of seismic data
- Session 4
  - Bandelets NG

Segmented Bandelet Basis

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

- Fast bandelet transform ($O(N^2)$):
  - resampling, fast warped wavelet transform, bandeletization.

Segmented Bandelet Basis

- Image support segmented in regions with either
  - a bandelet basis with a vertically parallel flow,
  - a bandelet basis with a horizontally parallel flow,
  - a wavelet basis (isotropic regularity).

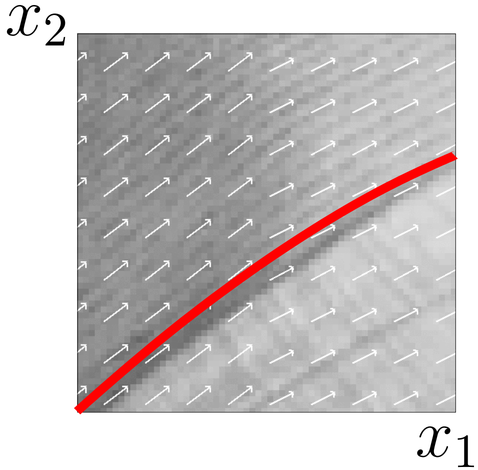
- Fast bandelet transform ($O(N^2)$):
  - resampling, fast warped wavelet transform, bandeletization.
- No blocking effect with an adapted lifting scheme.

Warped Wavelet Basis

Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$

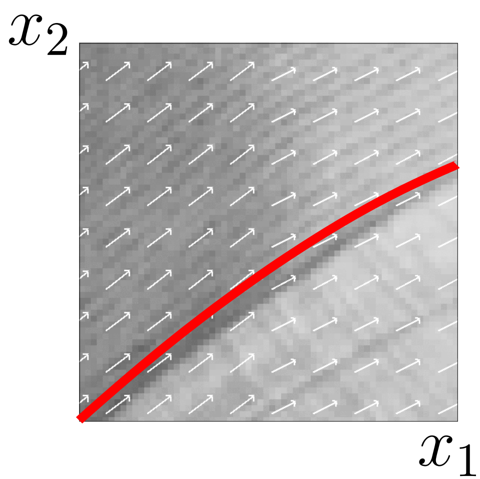


$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

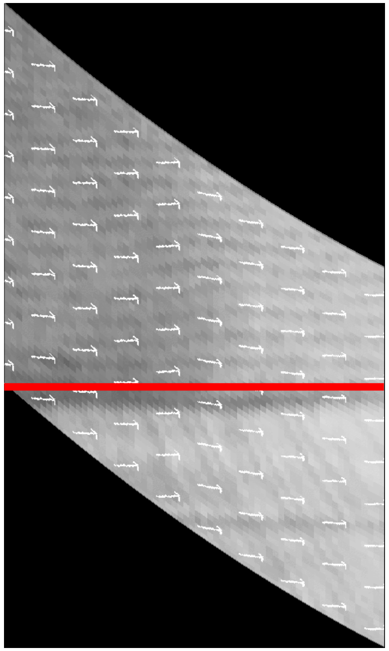
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\, \mathrm{d}u$$

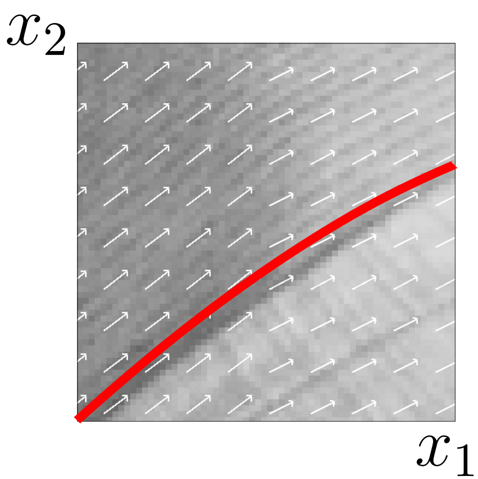


- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.

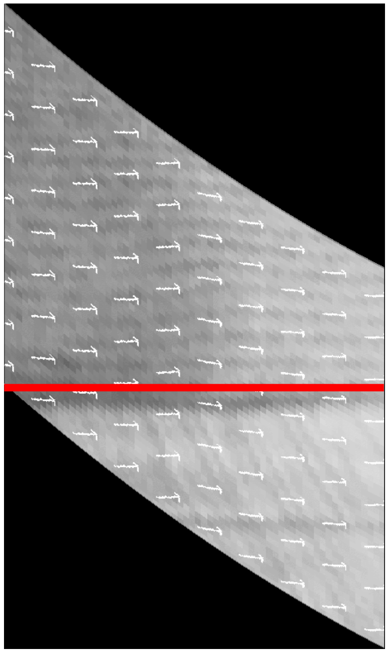
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

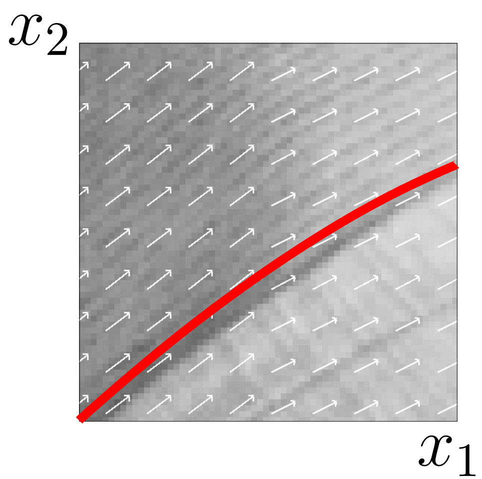


- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.
- $\langle f(x_1, x_2 + c(x_1))\,,\, \Psi(x_1, x_2)\rangle = \langle f(x_1, x_2)\,,\, \Psi(x_1, x_2 - c(x_1))\rangle$

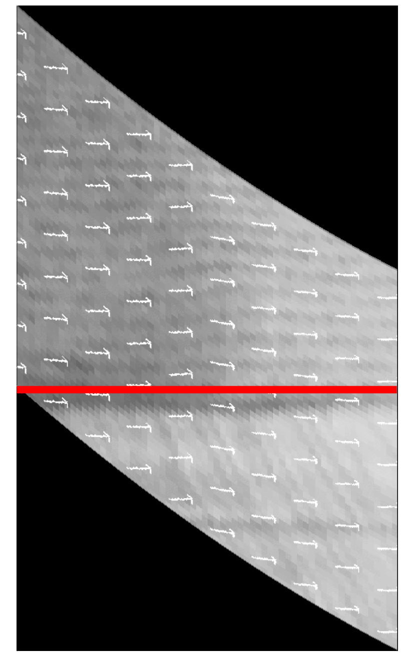
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$



- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.
- $\langle f(x_1, x_2 + c(x_1))\,,\, \Psi(x_1, x_2)\rangle = \langle f(x_1, x_2)\,,\, \Psi(x_1, x_2 - c(x_1))\rangle$
- Decomposition in a *warped wavelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}.$$

Bandeletization

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).

Bandeletization

- Basis function should have vanishing moments along x_1 (flow direction).
- Bandeletization: replace $\{\phi_{j,m_1}(x_1)\}_{m_1}$ with a wavelet family $\{\psi_{l,m_1}(x_1)\}_{l>j\,,m_1}$ that spans the same space.
- Warped wavelet basis of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}_j$$

Bandeletization

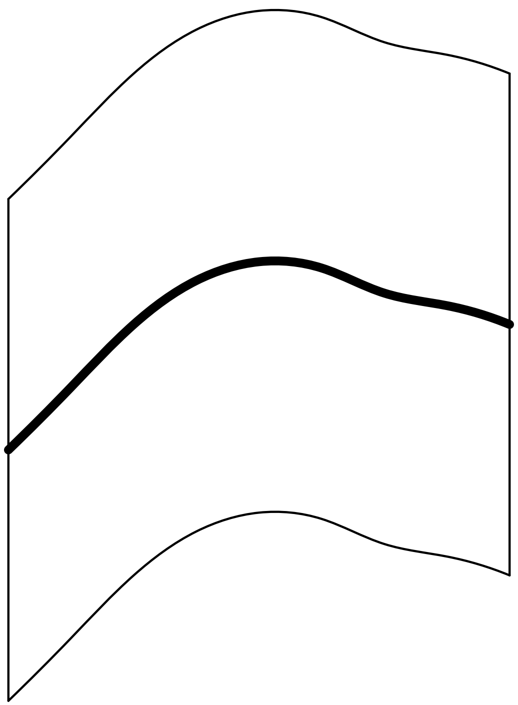
- Basis function should have vanishing moments along x_1 (flow direction).
- Bandeletization: replace $\{\phi_{j,m_1}(x_1)\}_{m_1}$ with a wavelet family $\{\psi_{l,m_1}(x_1)\}_{l>j\,,m_1}$ that spans the same space.
- *Bandelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \psi_{l,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}_{\substack{j,l>j \\ m_1,m_2}} .$$

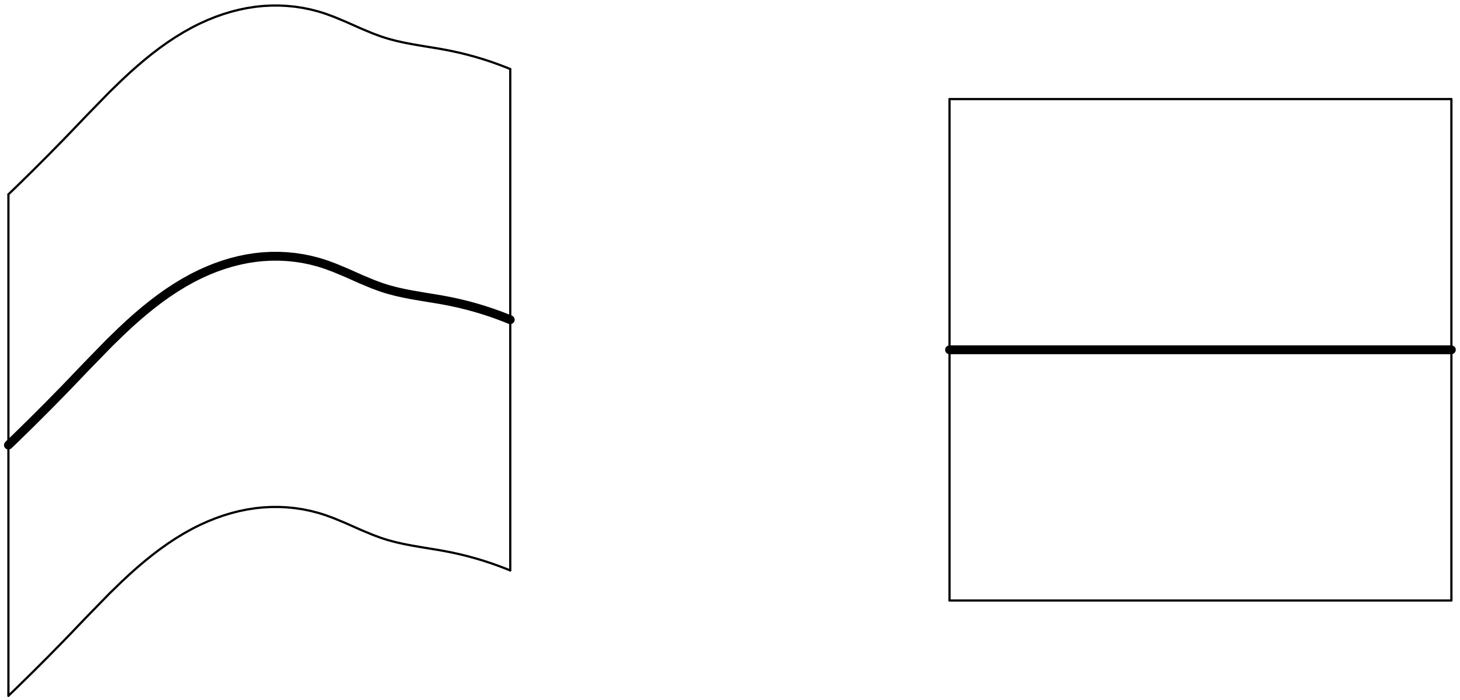
Anisotropic

Warped Wavelet Transform on a Tube

Warped Wavelet Transform on a Tube

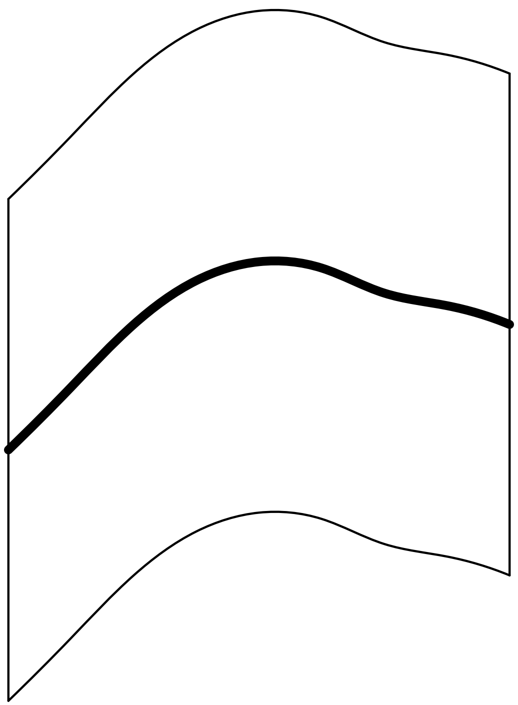


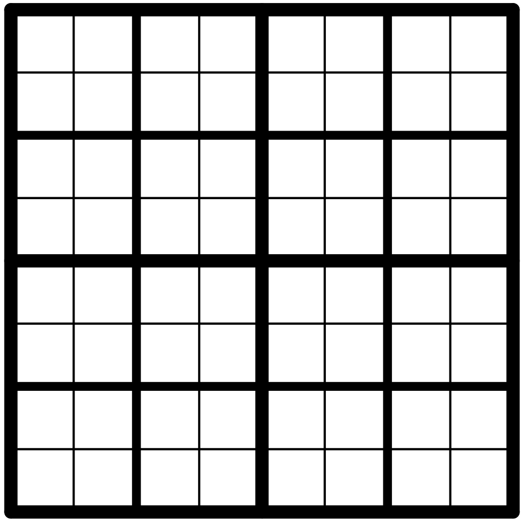
Warped Wavelet Transform on a Tube



- Simple setting: Tube, the natural structure associated to a flow.
- Warping to a rectangle (orthonormal transform).

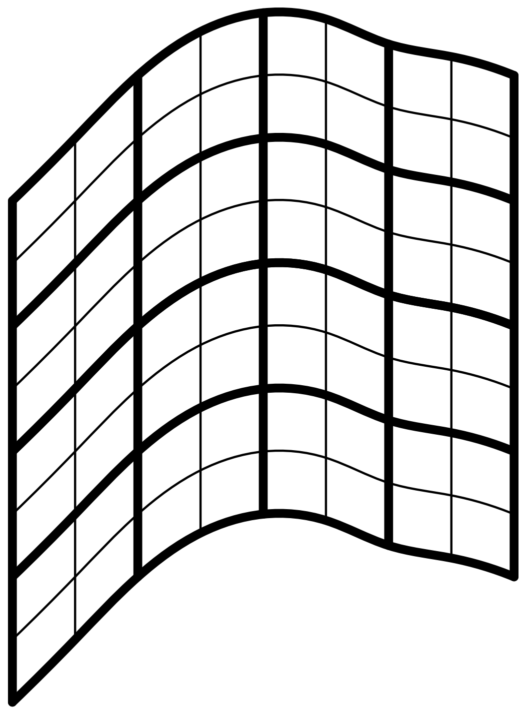
Warped Wavelet Transform on a Tube

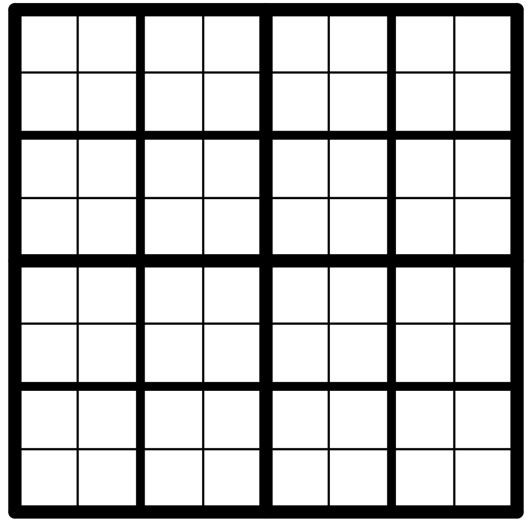




- Simple setting: Tube, the natural structure associated to a flow.
- Warping to a rectangle (orthonormal transform).
- Fast Wavelet Transform ($O(N^2)$).

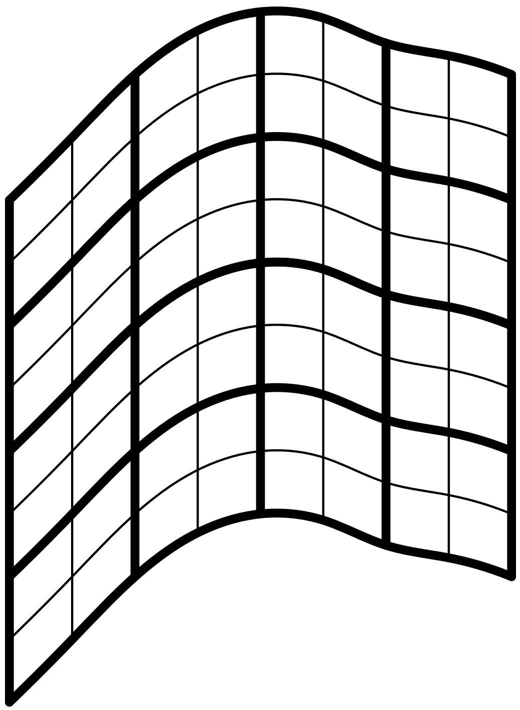
Warped Wavelet Transform on a Tube

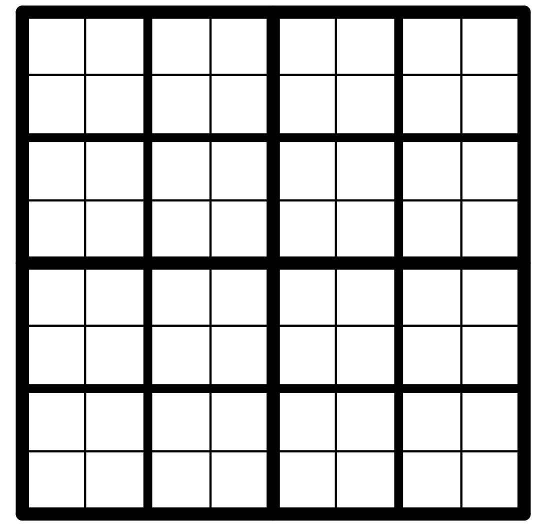




- Simple setting: Tube, the natural structure associated to a flow.
- Warping to a rectangle (orthonormal transform).
- Fast Wavelet Transform ($O(N^2)$).
- Warped Wavelet in the original domain.

Warped Wavelet Transform on a Tube





- Simple setting: Tube, the natural structure associated to a flow.
- Warping to a rectangle (orthonormal transform).
- Fast Wavelet Transform ($O(N^2)$).
- Warped Wavelet in the original domain.
- Inverse transform.

Bandeletization

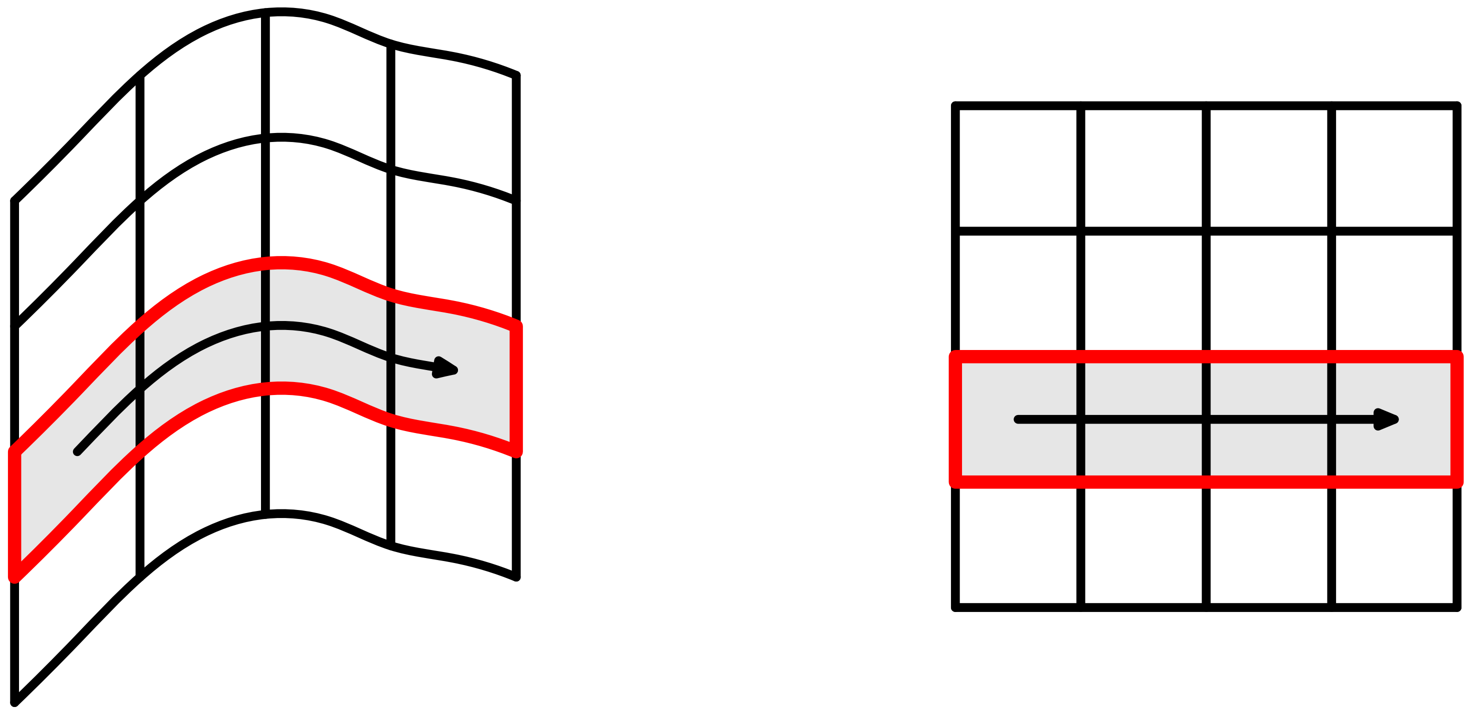
Bandeletization

Bandeletization

Bandeletization

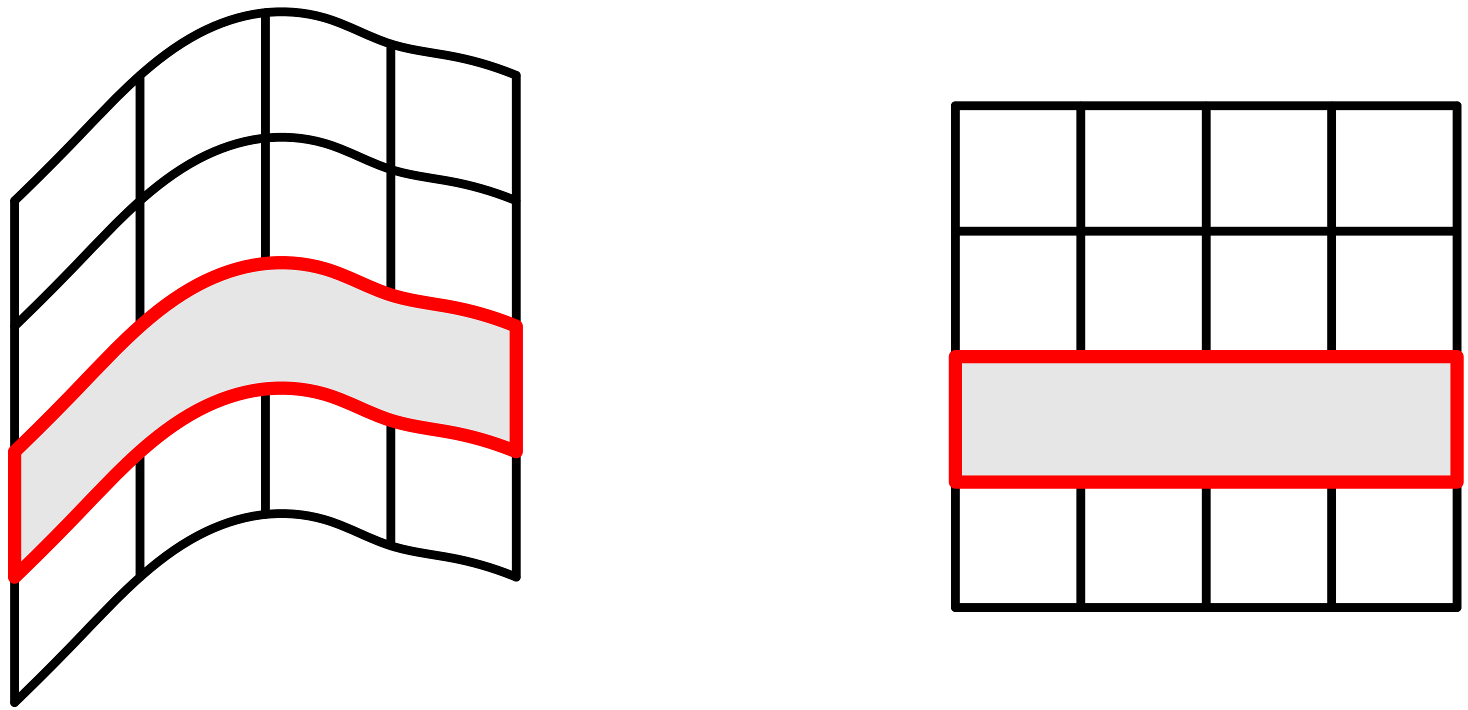
Bandeletization

Bandeletization



- Regularity along the geometry: regularity along the horizontal in the warped domain.
- 1D Wavelet Transform ($O(N^2)$):
  $\phi_{j,k_1}(x_1)\psi_{j,k_2} \to \psi_{l,k}(x_1)\psi_{j,k_2}(x_2)$.
- Bandelets: images of these hyperbolic wavelets in the original domain.

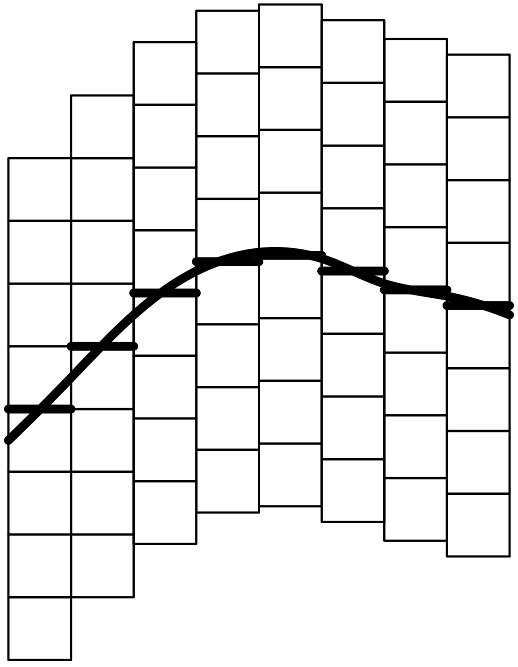
Bandeletization



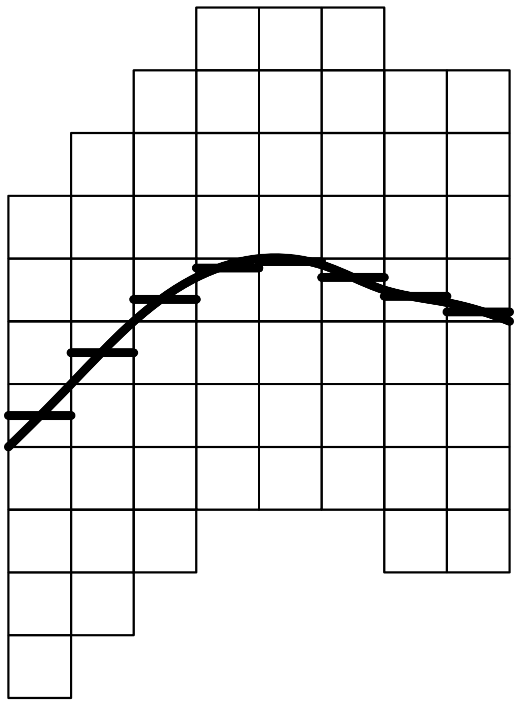
- Regularity along the geometry: regularity along the horizontal in the warped domain.
- 1D Wavelet Transform ($O(N^2)$): $\phi_{j,k_1}(x_1)\psi_{j,k_2} \to \psi_{l,k}(x_1)\psi_{j,k_2}(x_2)$.
- Bandelets: images of these hyperbolic wavelets in the original domain.

Discretization

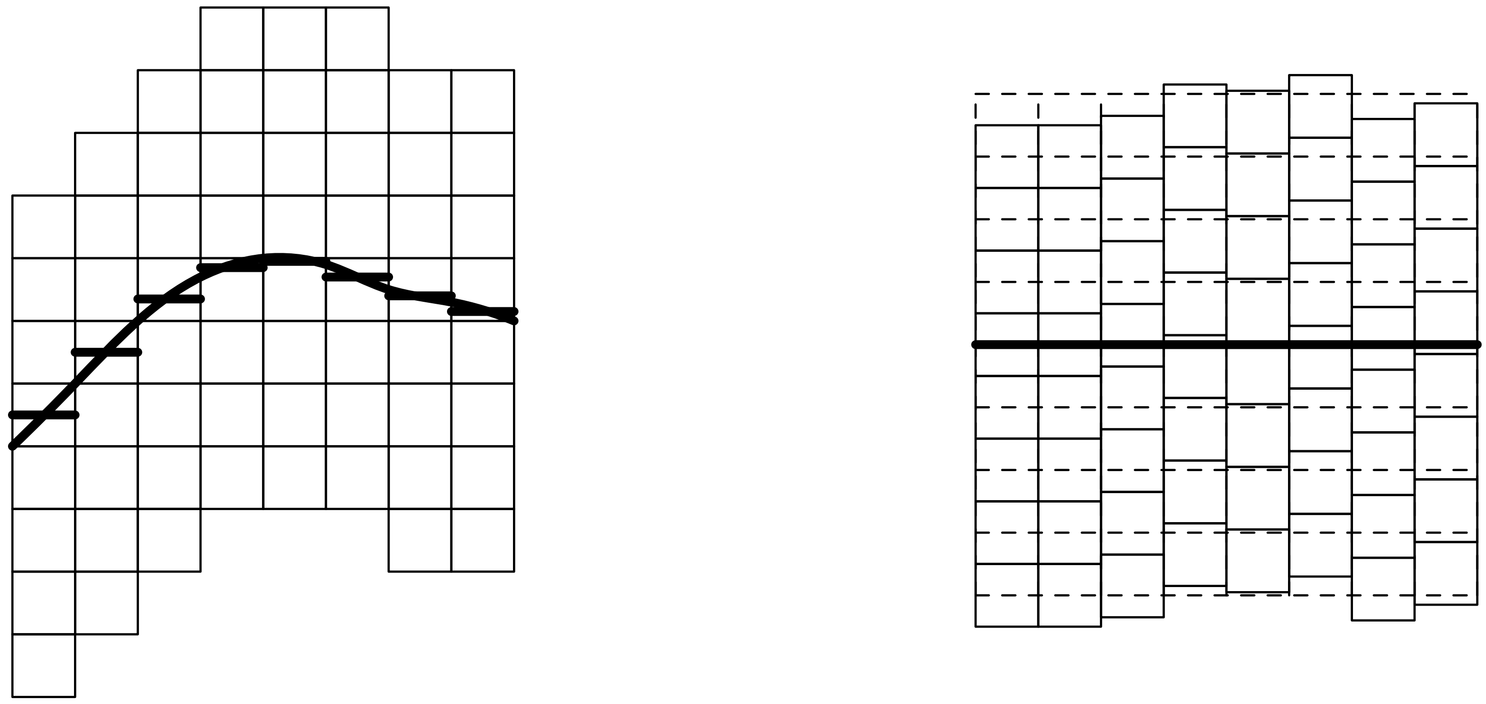
Discretization



Discretization

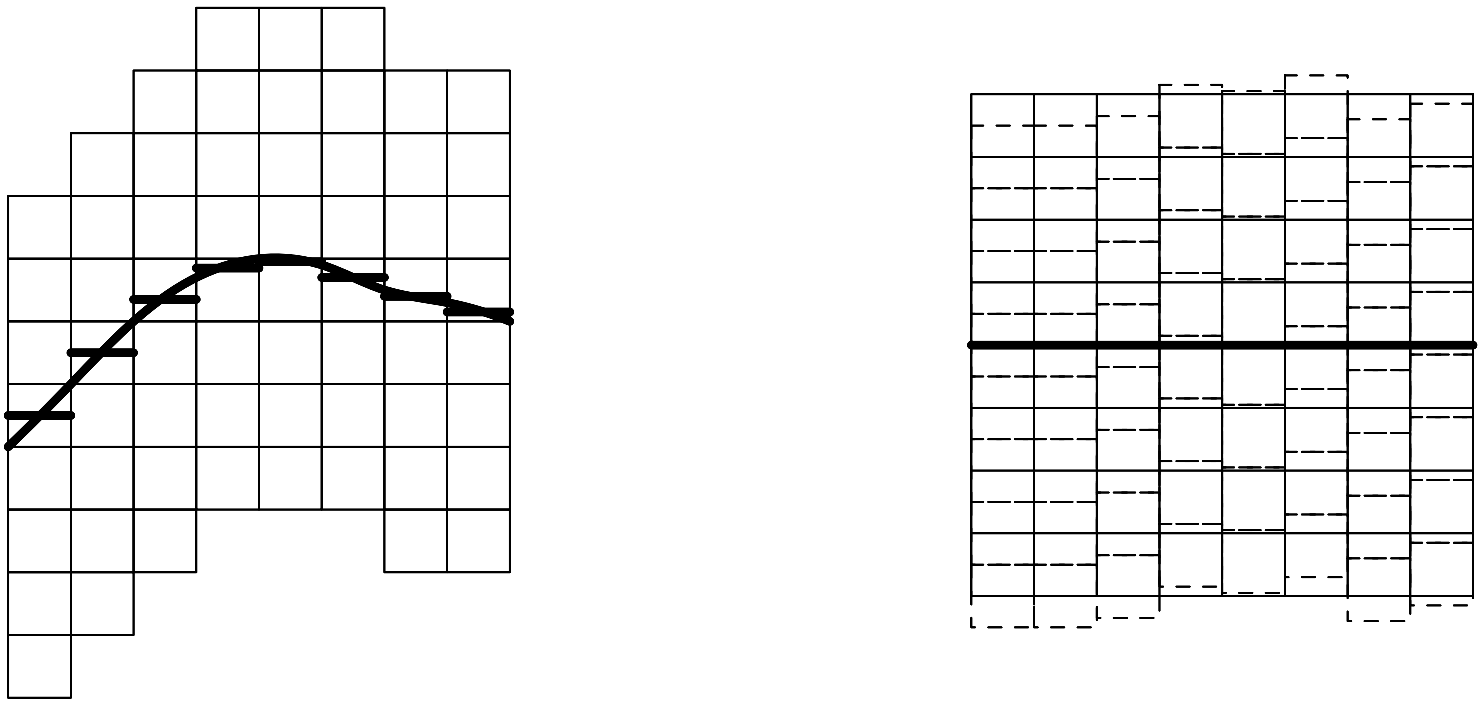


Discretization



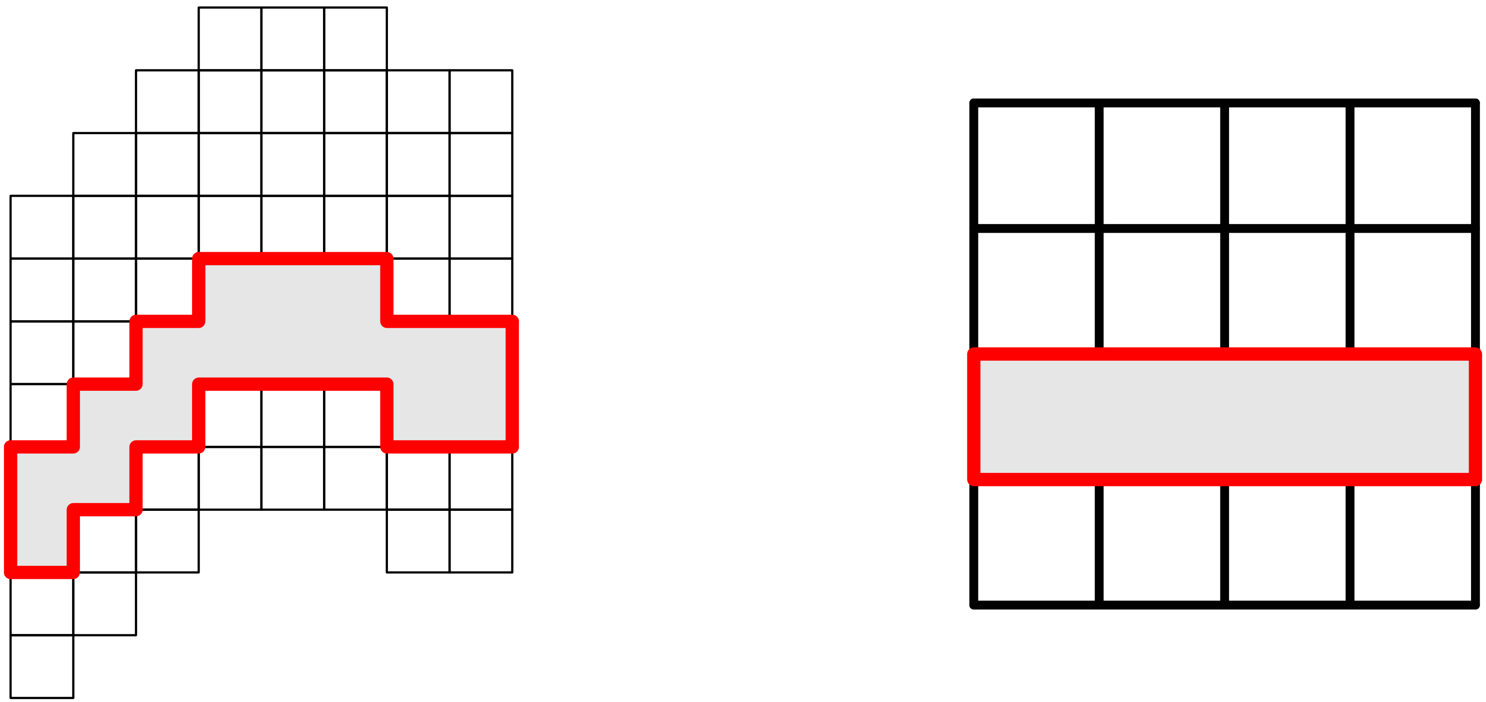
- Discretization : not adapted to the geometry.
- Resampling needed for the warping (choice very important).

Discretization



- Discretization : not adapted to the geometry.
- Resampling needed for the warping (choice very important).
- Fast Bandelet Transform: complexity $O(N^2)$.

Discretization

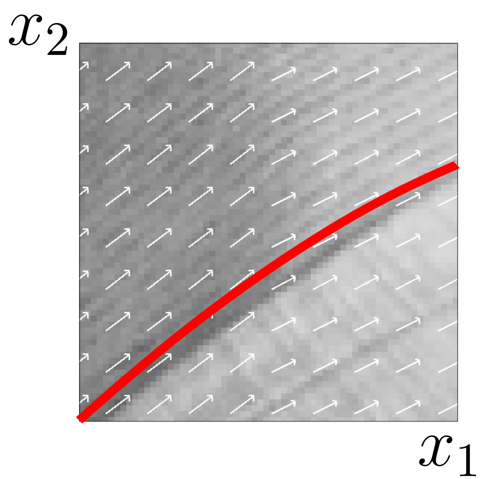


- Discretization : not adapted to the geometry.
- Resampling needed for the warping (choice very important).
- Fast Bandelet Transform: complexity $O(N^2)$.
- Interpolation of order 0: basis with a perfect reconstruction.

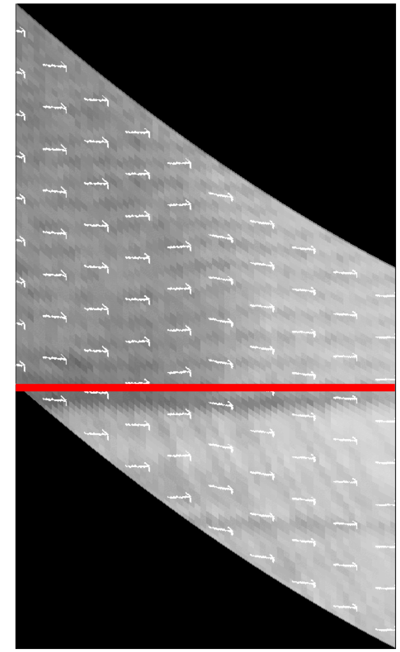
Warped Wavelet Basis

- Let the flow be vertically parallel:

$$\vec{\tau}(x_1, x_2) = (1\,,\, c'(x_1)).$$



$$c(x_1) = \int_{x_{1,\min}}^{x_1} c'(u)\,\mathrm{d}u$$

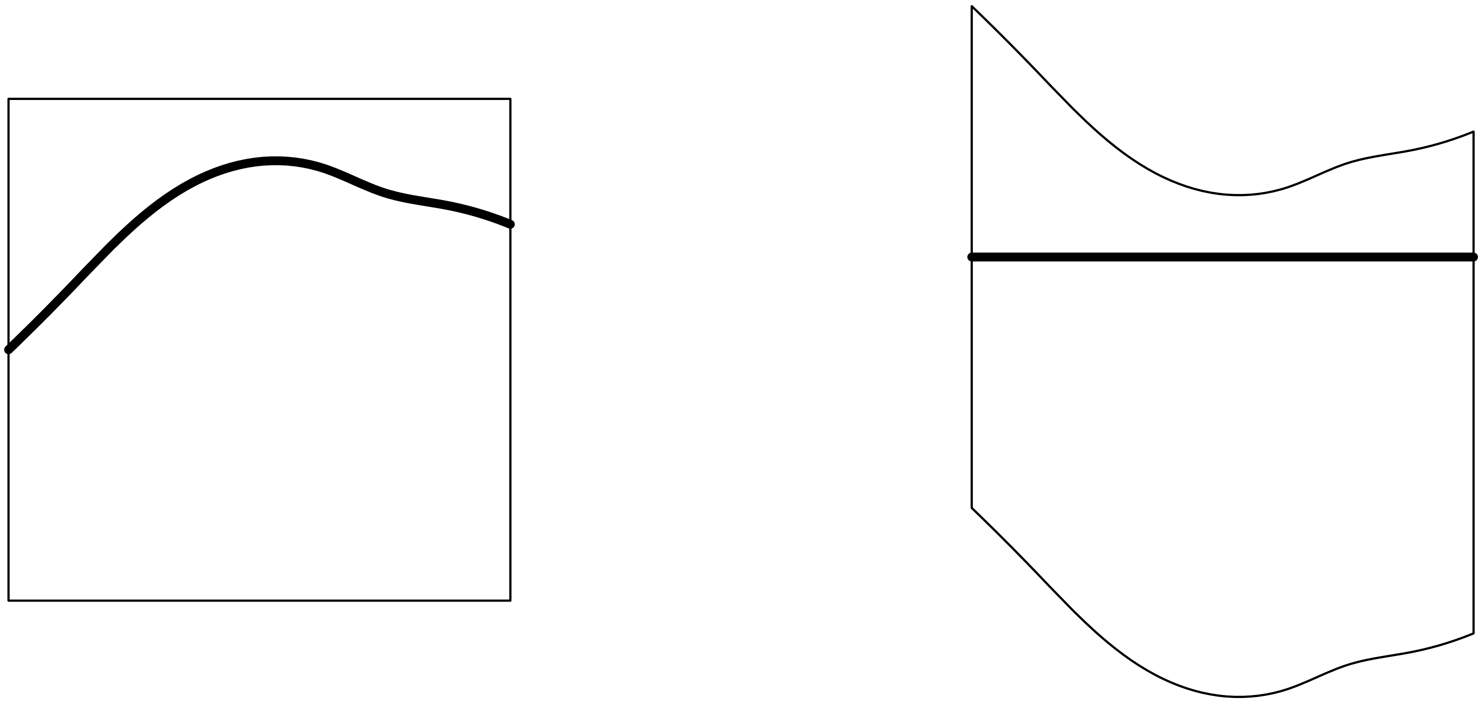


- For a given x_2, $f(x_1, x_2 + c(x_1))$ is a regular function of x_1.
- $\langle f(x_1, x_2 + c(x_1))\,,\, \Psi(x_1, x_2)\rangle = \langle f(x_1, x_2)\,,\, \Psi(x_1, x_2 - c(x_1))\rangle$
- Decomposition in a *warped wavelet basis* of $L^2(\Omega)$:

$$\left\{ \begin{array}{ccc} \phi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) & , & \psi_{j,m_1}(x_1)\,\phi_{j,m_2}(x_2 - c(x_1)) \\ & , & \psi_{j,m_1}(x_1)\,\psi_{j,m_2}(x_2 - c(x_1)) \end{array} \right\}.$$

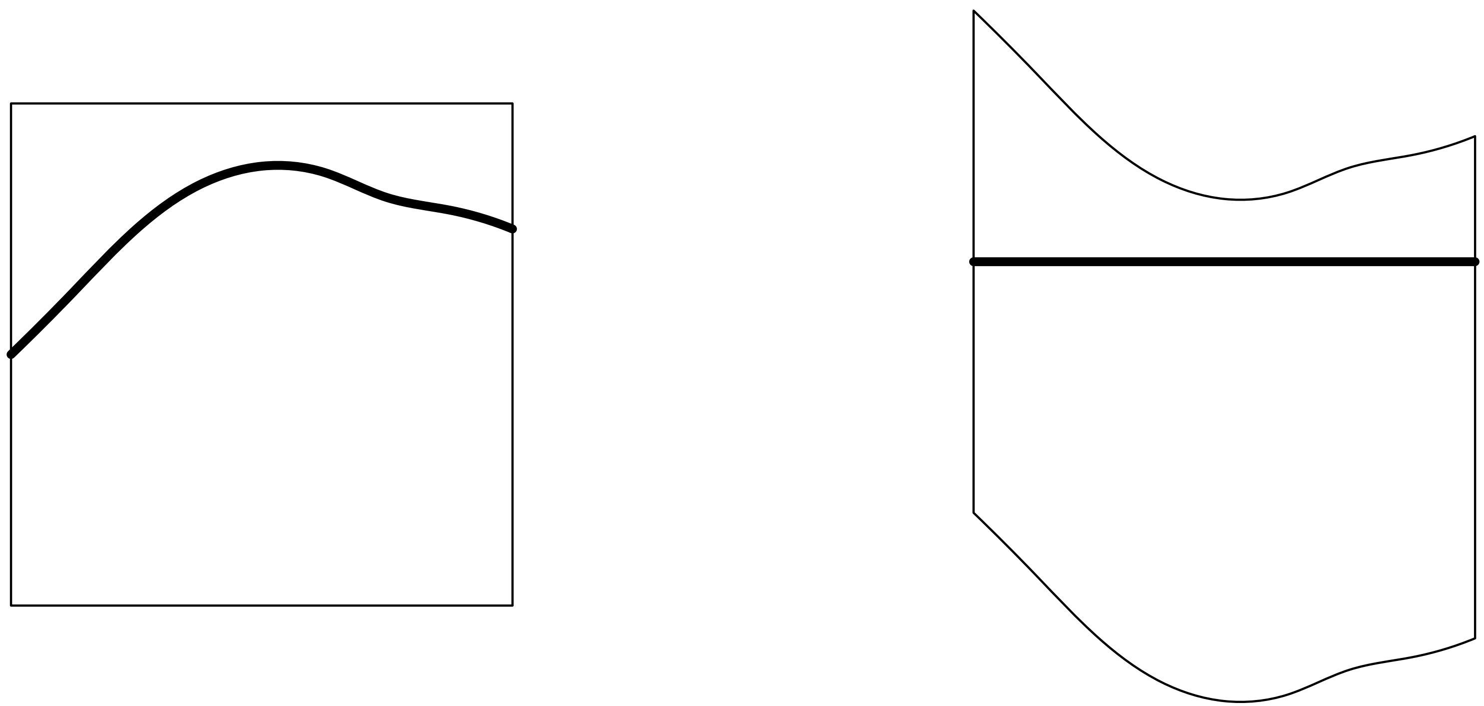
Bandelets on a Square

Bandelets on a Square



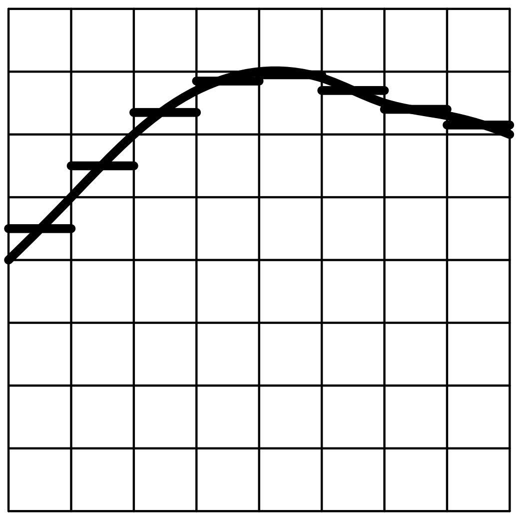
- Wavelets on an arbitrary domain.

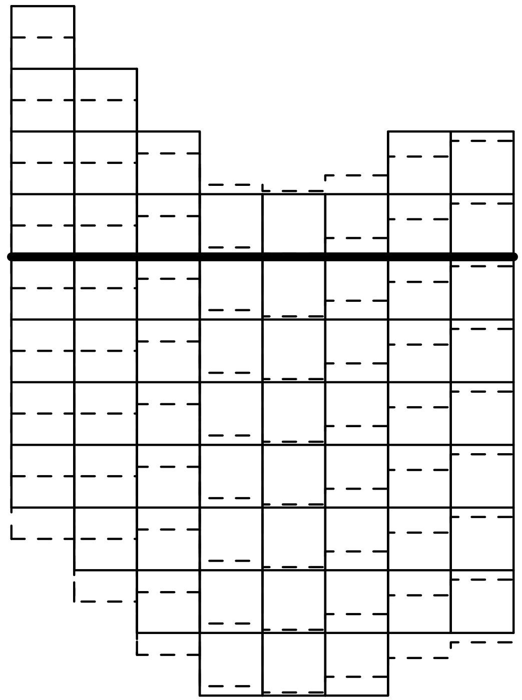
Bandelets on a Square



- Wavelets on an arbitrary domain.
- Continuous case: existence of a warped wavelet basis.

Bandelets on a Square

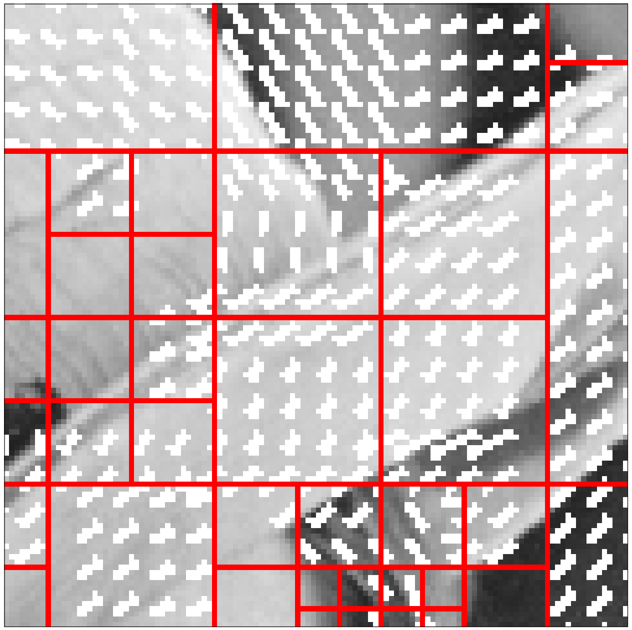




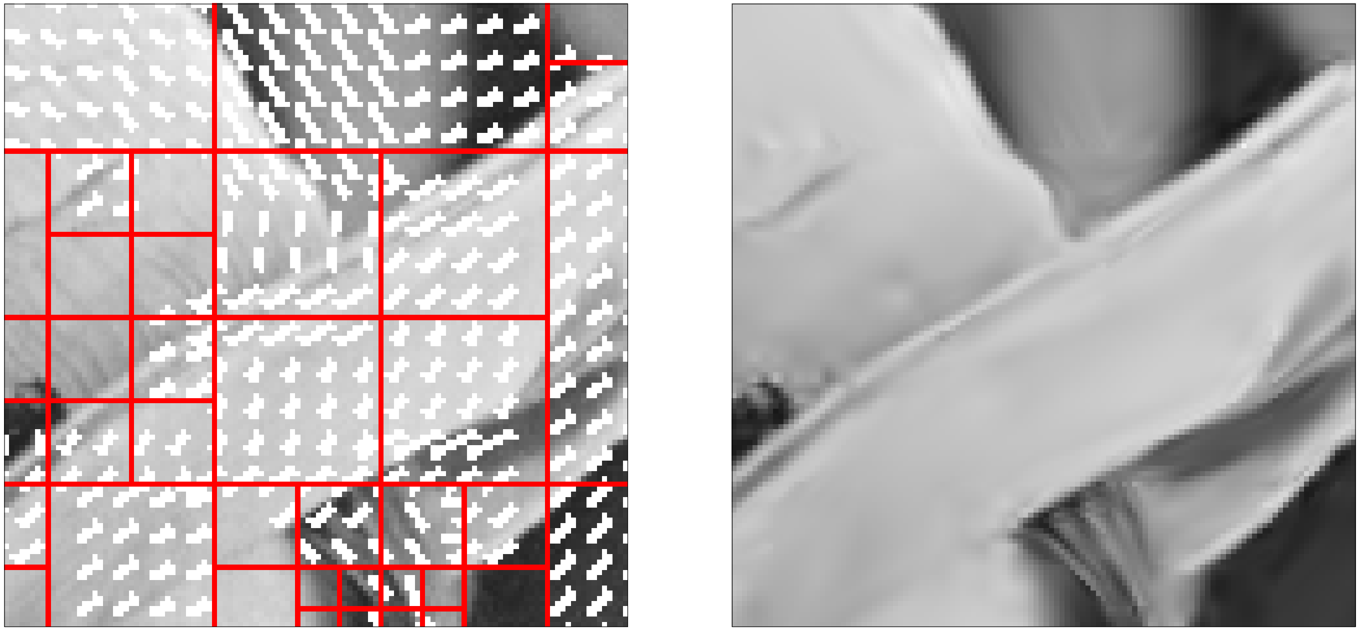
- Wavelets on an arbitrary domain.
- Continuous case: existence of a warped wavelet basis.
- Discrete case: Lifting scheme.

Blocking Effect

Blocking Effect

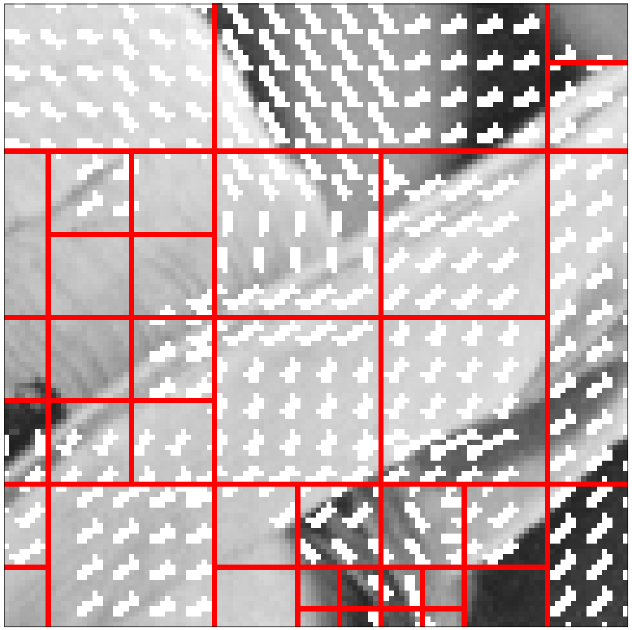


Blocking Effect



- Quadtree segmentation + basis on each square = blocking artifacts.
- But not in practice!

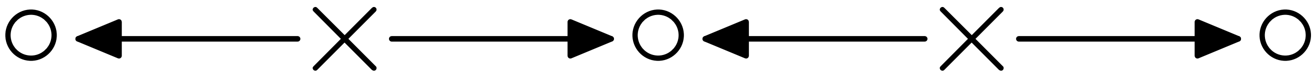
Blocking Effect



- Quadtree segmentation + basis on each square = blocking artifacts.
- But not in practice!
- Modification of the classical wavelet transform: Lifting Scheme.

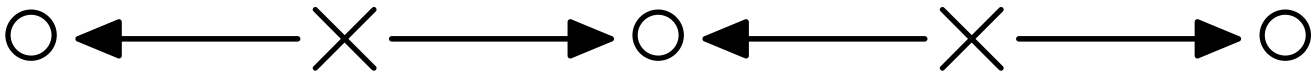
Lifting Scheme

Lifting Scheme



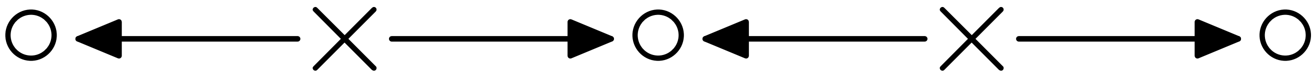
- Lifting scheme: versatile way to implement a wavelet transform and more...

Lifting Scheme



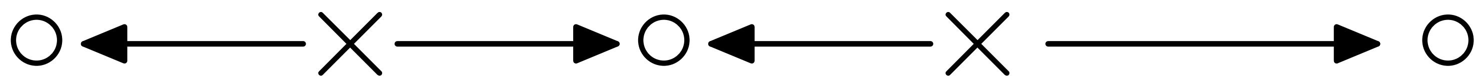
- Lifting scheme: versatile way to implement a wavelet transform and more...
- Basic steps:
  - *predict* the odd samples with the even samples.
  - *update* the even samples with the new odd samples.
  - iterate the two previous steps.
  - *rescale* the resulting samples.

Lifting Scheme



- Lifting scheme: versatile way to implement a wavelet transform and more...
- Basic steps:
  - *predict* the odd samples with the even samples.
  - *update* the even samples with the new odd samples.
  - iterate the two previous steps.
  - *rescale* the resulting samples.
- Only requires a neighborhood of size 1!

Lifting Scheme

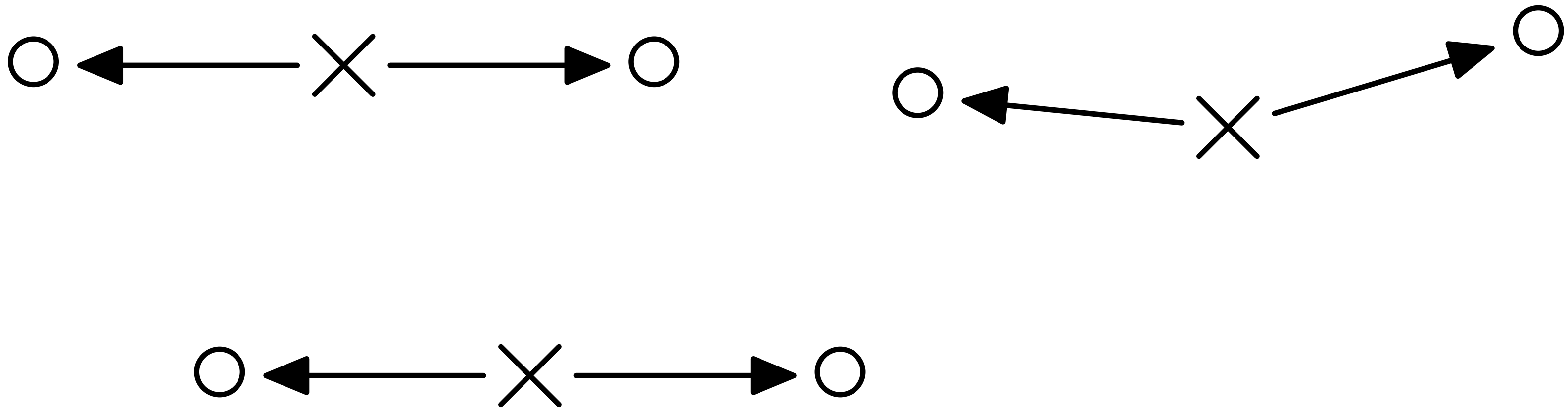


- Lifting scheme: versatile way to implement a wavelet transform and more...
- Basic steps:
  - *predict* the odd samples with the even samples.
  - *update* the even samples with the new odd samples.
  - iterate the two previous steps.
  - *rescale* the resulting samples.
- Only requires a neighborhood of size 1!
- Easy for irregular grid in the symetric case (still 2 vanishing moments).

Bandelet Lifting Scheme

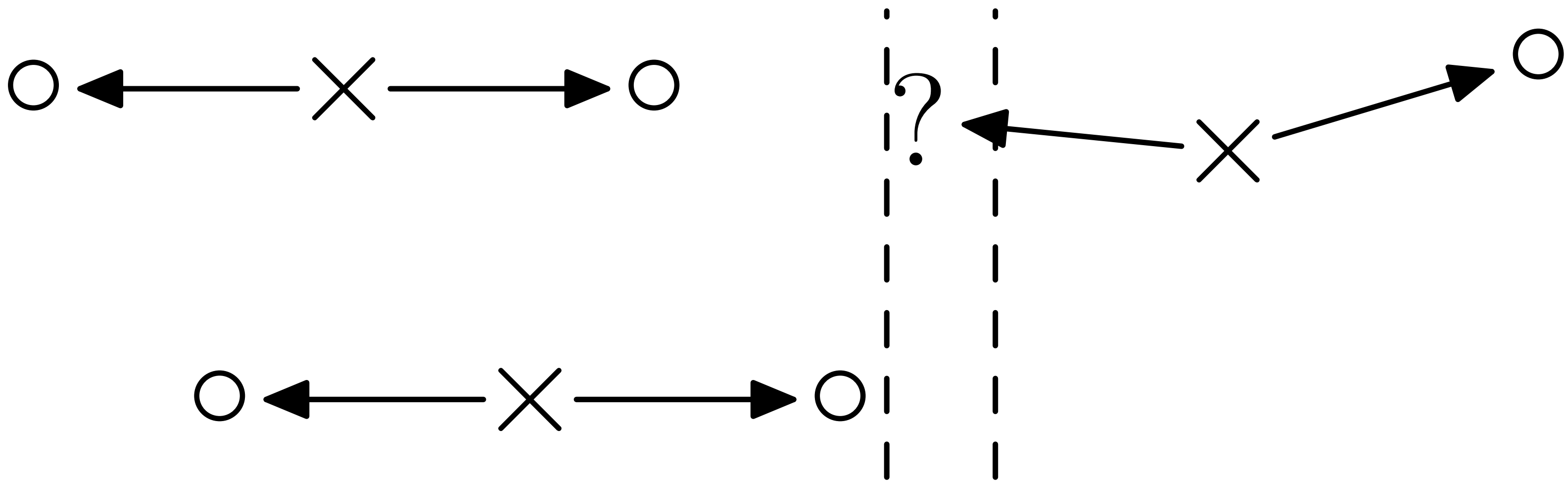
Bandelet Lifting Scheme

Bandelet Lifting Scheme



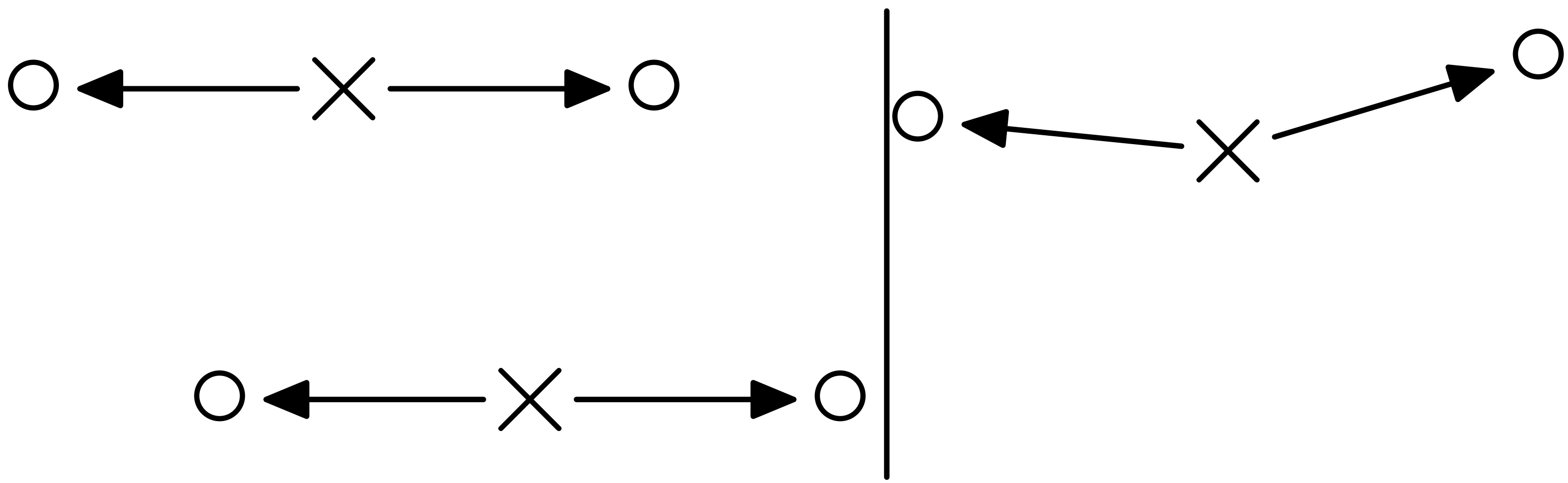
- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.

Bandelet Lifting Scheme



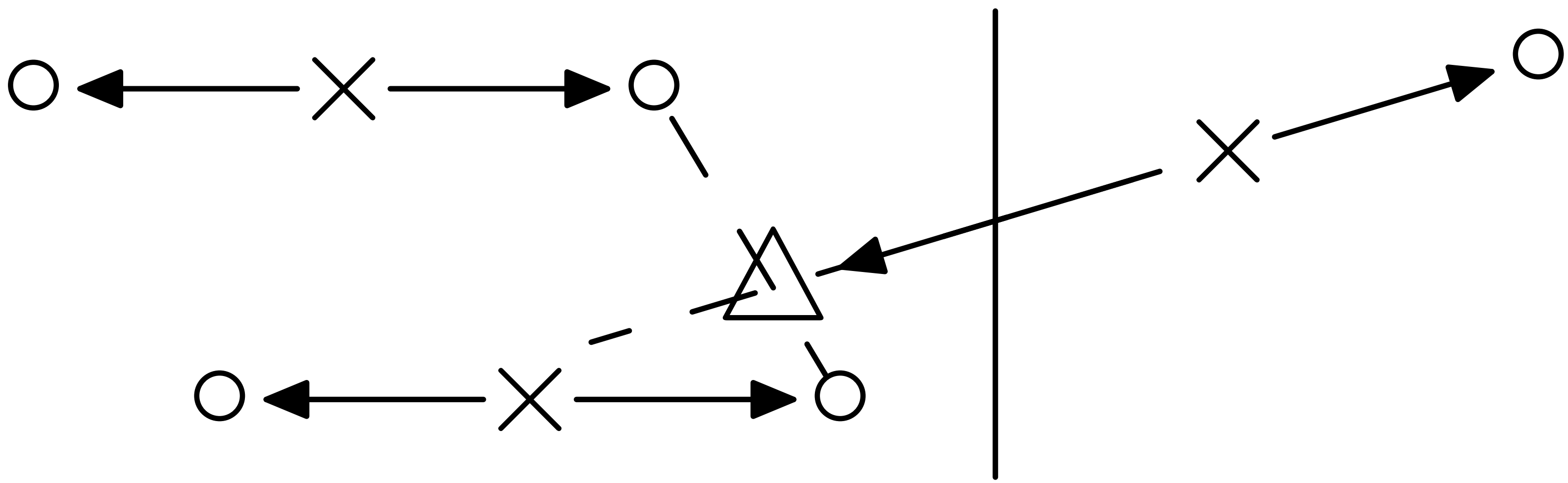
- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.
- Only difference with the regular lifting scheme is at the boundaries.

Bandelet Lifting Scheme



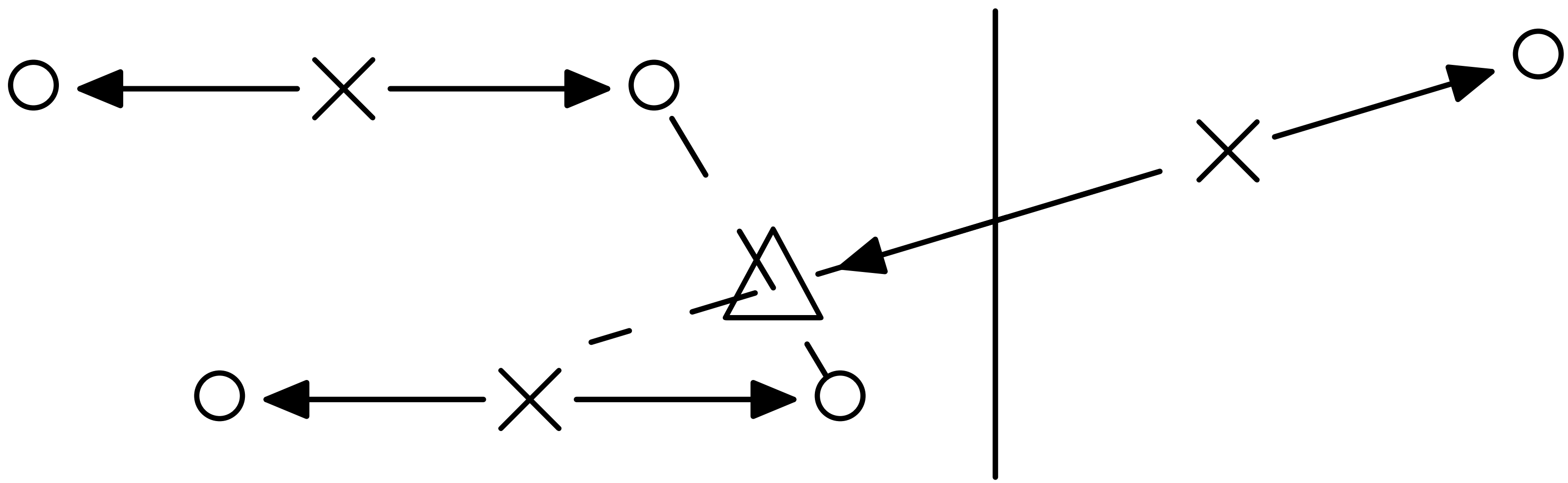
- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.
- Only difference with the regular lifting scheme is at the boundaries.

Bandelet Lifting Scheme



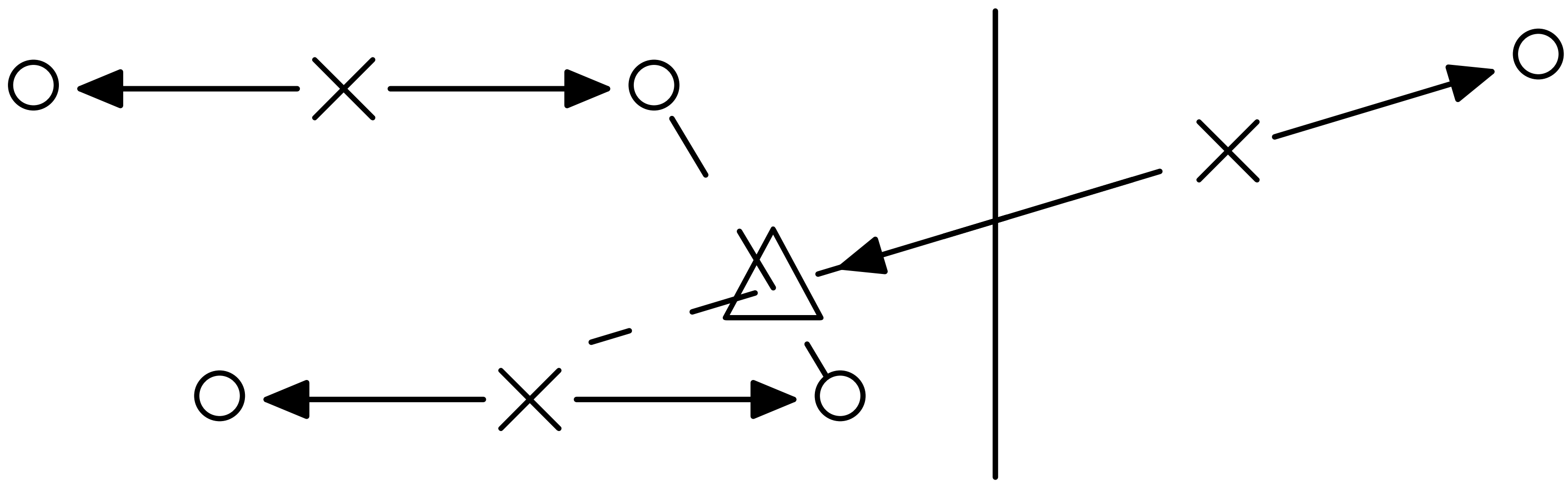
- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.
- Only difference with the regular lifting scheme is at the boundaries.

Bandelet Lifting Scheme



- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.
- Only difference with the regular lifting scheme is at the boundaries.
- Still 2 vanishing moments.

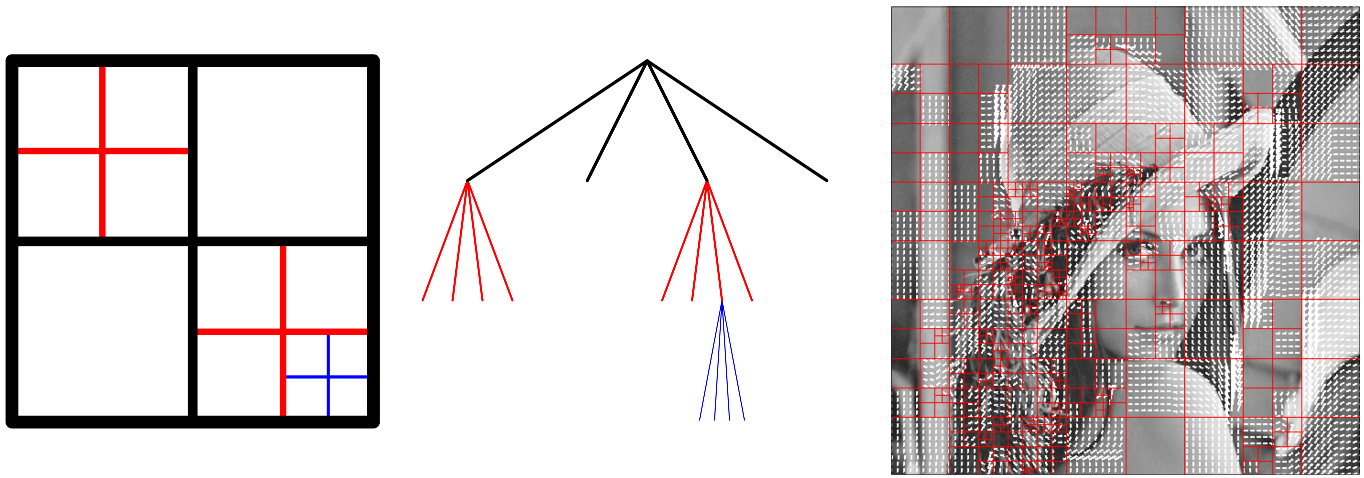
Bandelet Lifting Scheme



- 2D wavelet lifting scheme: extension of the 1D case (along lines and colums).
- Extension to an irregular sampling case.
- Only difference with the regular lifting scheme is at the boundaries.
- Still 2 vanishing moments.
- Bandelet lifting scheme: add another 1D transform.

CART Algorithm

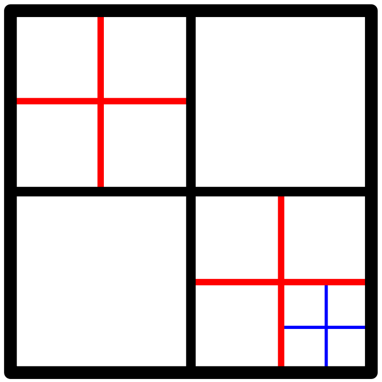
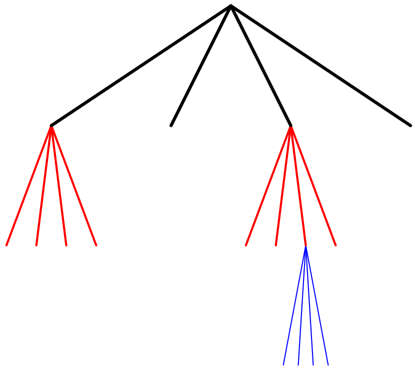
CART Algorithm



- Additive structure of the Lagrangian:

$$||f - f_M||^2 + T^2 M = \sum_i ||f - f_M||^2_{\Omega_i} + T^2 M_i \quad .$$

CART Algorithm

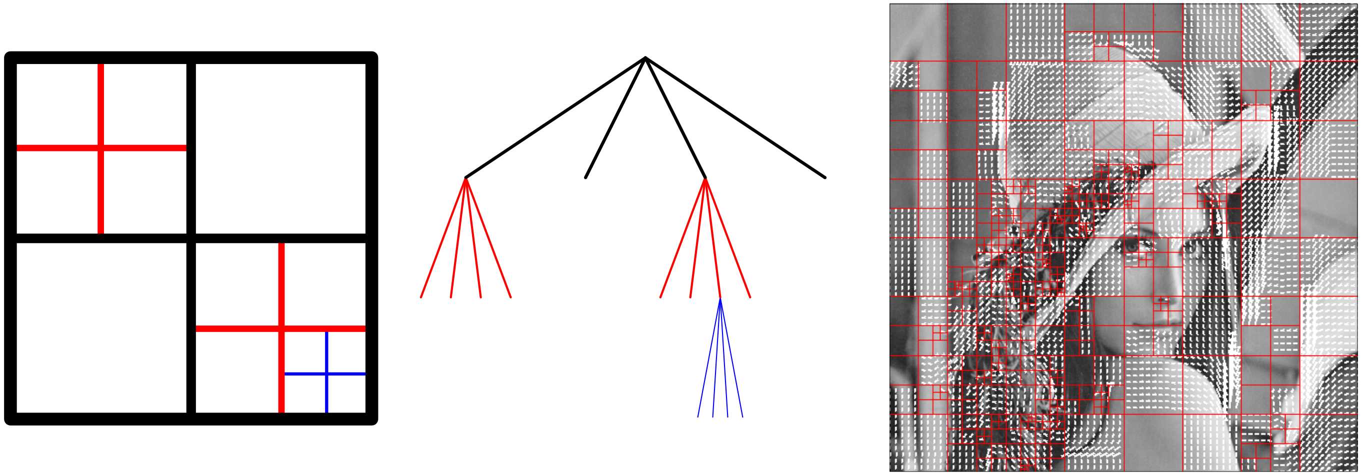




- Additive structure of the Lagrangian:

$$||f - f_M||^2 + T^2 M = \sum_i ||f - f_M||^2_{\Omega_i} + T^2 M_i \quad .$$

- Bottom to top dynamic programming.

CART Algorithm

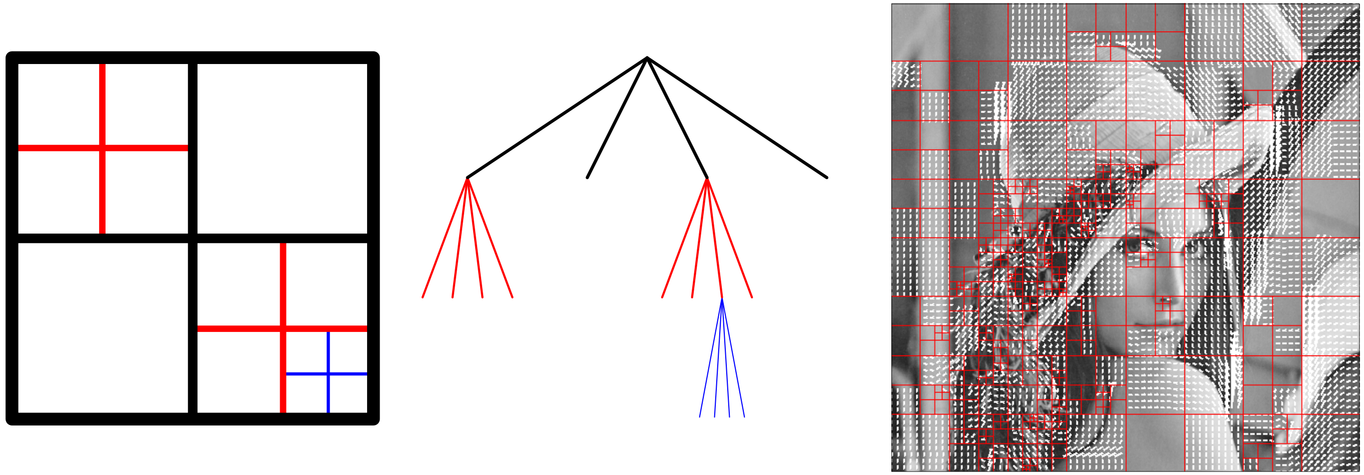


- Additive structure of the Lagrangian:

$$||f - f_M||^2 + T^2 M = \sum_i ||f - f_M||^2_{\Omega_i} + T^2 M_i \quad .$$

- Bottom to top dynamic programming.
- Minimization on each dyadic square.

CART Algorithm



- Additive structure of the Lagrangian:

$$||f - f_M||^2 + T^2 M = \sum_i ||f - f_M||^2_{\Omega_i} + T^2 M_i \quad .$$

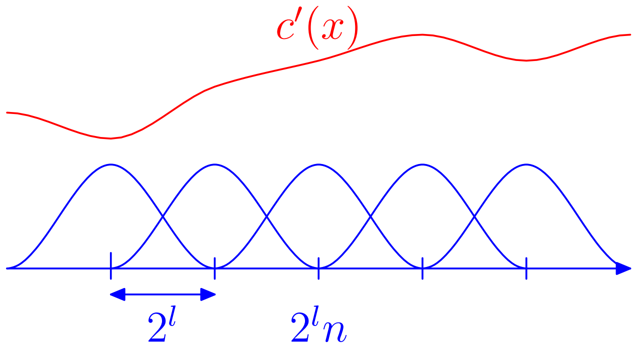
- Bottom to top dynamic programming.
- Minimization on each dyadic square.
- Global minimization on all the possible geometries.

Geometry Estimation

Geometry Estimation

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$

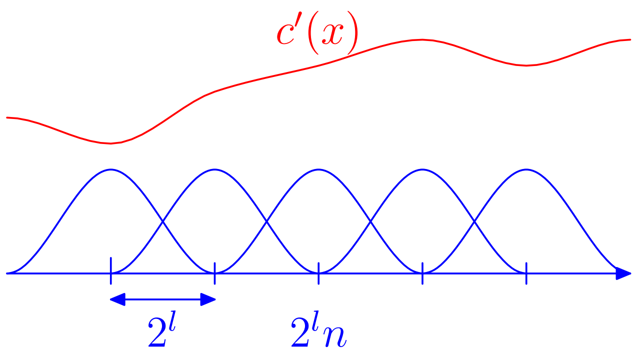


and the $L\,2^{-l}$ parameters α_n.

Geometry Estimation

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$



and the $L\,2^{-l}$ parameters α_n.

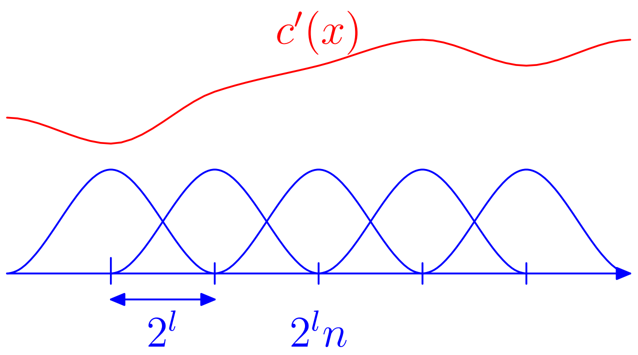
- Minimization of

$$\int_\Omega \left| \nabla f(x_1, x_2) \, . \, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \, .$$

Geometry Estimation

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n)$$



and the $L\,2^{-l}$ parameters α_n.

- Minimization of

$$\int_\Omega \left| \nabla f(x_1, x_2) \, . \, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \, .$$

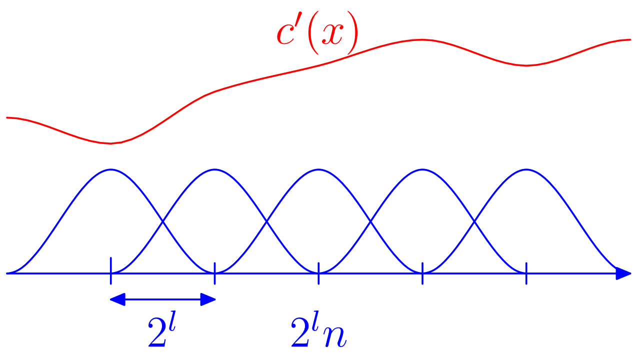
- Lucas-Kanade : minimization of

$$\sum_\Omega |f[n_1, n_2] - f[n_1 + 1, n_2 + c'(n_1)]|^2$$

Geometry Estimation

- A vertically parallel flow $\vec{\tau}(x_1, x_2) = (1, c'(x_1))$ in Ω is parameterized by

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l} x - n)$$



and the $L\,2^{-l}$ parameters α_n.

- Minimization of

$$\int_\Omega \left| \nabla f(x_1, x_2) \, . \, \vec{\tau}(x_1, x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 = \int_\Omega \left| \frac{\partial f(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \, .$$

- Lucas-Kanade : minimization of

$$\sum_\Omega |f[n_1, n_2] - f[n_1 + 1, n_2 + c'(n_1)]|^2$$

- Closely related to the optical flow

Computational Issues

Computational Issues

- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.

Computational Issues

- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.
- Estimation of a geometry per square: complexity $O(N^2 \log(N)^2)$.

Computational Issues

- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.
- Estimation of a geometry per square: complexity $O(N^2 \log(N)^2)$.
- Test all geometries:
  $c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$

Computational Issues

- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.
- Estimation of a geometry per square: complexity $O(N^2 \log(N)^2)$.
- Test all geometries:
  $c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$
- Exponential complexity!

Computational Issues

- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.
- Estimation of a geometry per square: complexity $O(N^2 \log(N)^2)$.
- Test all geometries: $c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$
- Exponential complexity!
- Polynomial geometry: somewhere in between?

Computational Issues

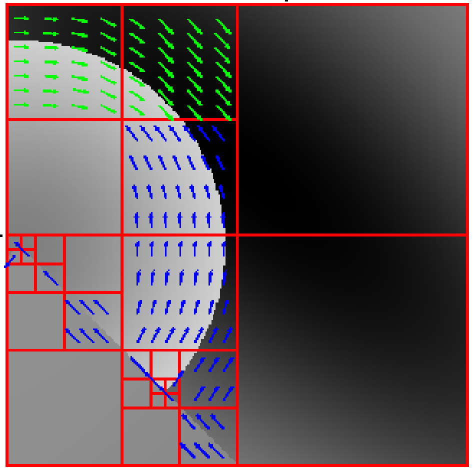
- Cost of the minimization on each dyadic square depends on the possible choices for the geometries.
- Estimation of a geometry per square: complexity $O(N^2 \log(N)^2)$.
- Test all geometries:
  $c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$
- Exponential complexity!
- Polynomial geometry: somewhere in between?
- Free lunch: Much better for the estimation setting (non local).

Theorem

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

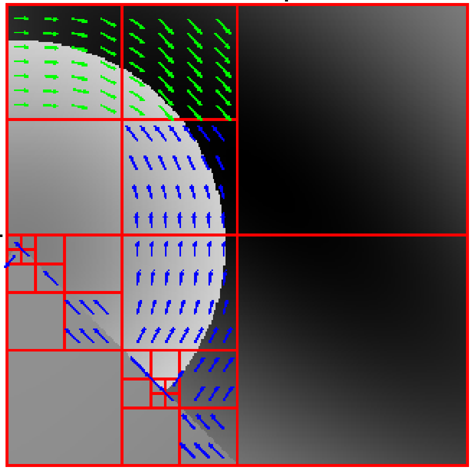
$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$

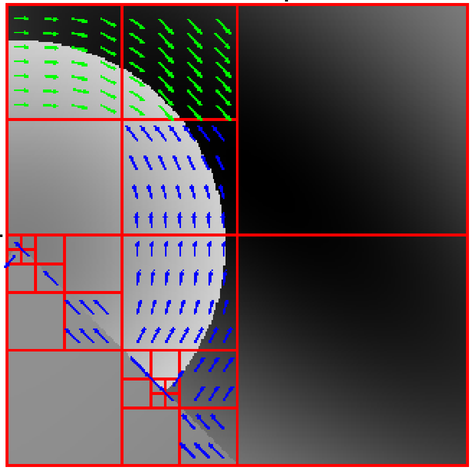


- Unknown degree of smoothness α.

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$

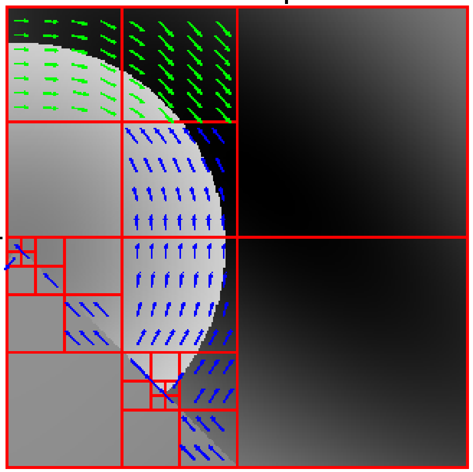


- Unknown degree of smoothness α.
- Optimal decay exponent α.

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



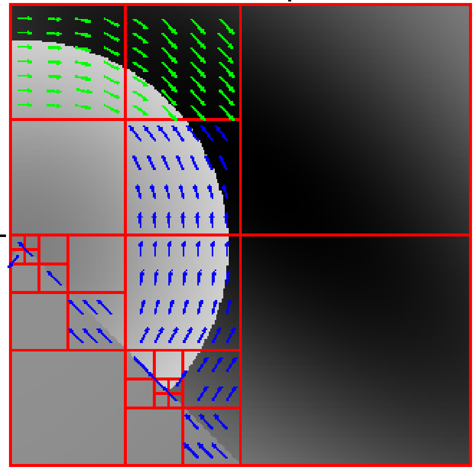
- Unknown degree of smoothness α.
- Optimal decay exponent α.
- Improvement over isotropic wavelets for which

$$\|f - f_M\|^2 \leqslant C\, M^{-1} \quad .$$

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \;.$$



- Unknown degree of smoothness α.
- Optimal decay exponent α.
- Improvement over isotropic wavelets for which

$$\|f - f_M\|^2 \leqslant C\, M^{-1} \quad .$$

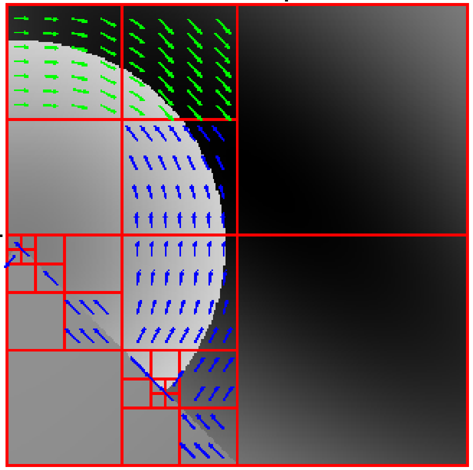
- Improvement over curvelets for which

$$\|f - f_M\|^2 \leqslant C\, (\log M)^3\, M^{-2} \quad .$$

Theorem

- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ ou $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$, with some non tangency conditions)

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



- Unknown degree of smoothness α.
- Optimal decay exponent α.
- Improvement over isotropic wavelets for which

$$\|f - f_M\|^2 \leqslant C\, M^{-1} \quad .$$

- Improvement over curvelets for which

$$\|f - f_M\|^2 \leqslant C\, (\log M)^3\, M^{-2} \quad .$$

- **But** with slightly modified bandelets.

Bandelet Frame

Bandelet Frame

- Frame instead of basis:

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.
  - Bandelet basis associated to the smallest tube that comprises the square.

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.
  - Bandelet basis associated to the smallest tube that comprises the square.
- Choice of the geometry:

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.
  - Bandelet basis associated to the smallest tube that comprises the square.
- Choice of the geometry:
  - Estimation of the geometry possible only if the jumps of the discontinuities are high enough.

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.
  - Bandelet basis associated to the smallest tube that comprises the square.
- Choice of the geometry:
  - Estimation of the geometry possible only if the jumps of the discontinuities are high enough.
  - General case: exhaustive exploration of all possible geometries in a square up to a precision ϵ.

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$$

Bandelet Frame

- Frame instead of basis:
  - Orthonormal families of bandelets that overflow their associated square: redundancy.
  - Bandelet basis associated to the smallest tube that comprises the square.
- Choice of the geometry:
  - Estimation of the geometry possible only if the jumps of the discontinuities are high enough.
  - General case: exhaustive exploration of all possible geometries in a square up to a precision ϵ.

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n \, \phi(2^{-l}x - n) \quad \forall \alpha_n \in \epsilon\mathbb{Z} \cap [-2, 2]$$

  - Exponential complexity!

Polynomial flow

Polynomial flow

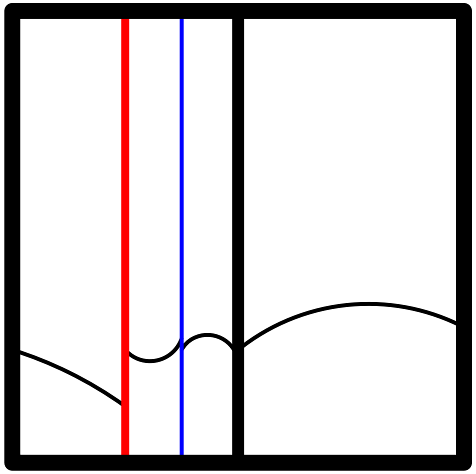
- Complexity reduction by a reduction of the number of the possible geometries.

Polynomial flow

- Complexity reduction by a reduction of the number of the possible geometries.
- Polynomial flow: $$c'(x) = \sum_{n=0}^{p} \alpha_n \, x^n \quad .$$

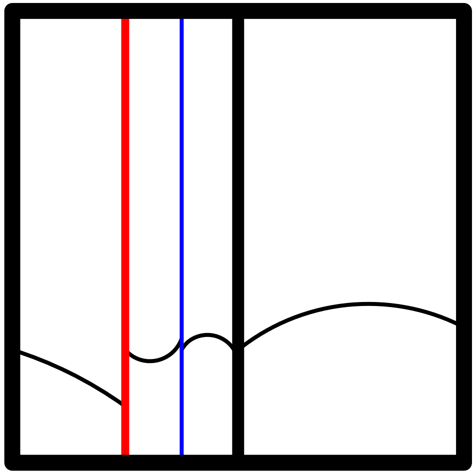
Polynomial flow

- Complexity reduction by a reduction of the number of the possible geometries.
- Polynomial flow: $$c'(x) = \sum_{n=0}^{p} \alpha_n \, x^n \quad .$$
- Further splitting of the dyadic squares in dyadic rectangles.



Polynomial flow

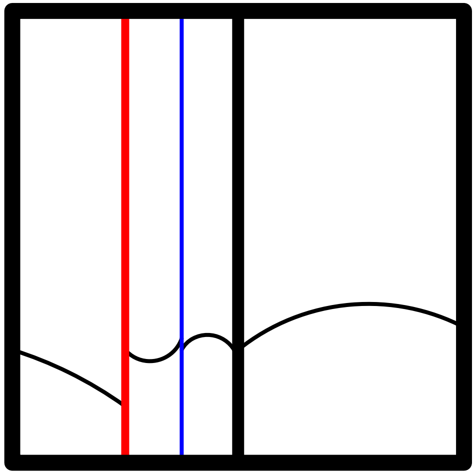
- Complexity reduction by a reduction of the number of the possible geometries.
- Polynomial flow: $$c'(x) = \sum_{n=0}^{p} \alpha_n \, x^n \quad .$$
- Further splitting of the dyadic squares in dyadic rectangles.



- Optimization algorithm with polynomial complexity.

Polynomial flow

- Complexity reduction by a reduction of the number of the possible geometries.
- Polynomial flow: $$c'(x) = \sum_{n=0}^{p} \alpha_n \, x^n \quad .$$
- Further splitting of the dyadic squares in dyadic rectangles.



- Optimization algorithm with polynomial complexity.
- Logarithmic factor in the decay:
$$\|f - f_M\|^2 \leqslant C M^{-\alpha} \log M$$

Theorem Proof

Theorem Proof

- Theshold T : Lagrangian $\mathcal{L}(f, T, \mathbf{B}) = \sum_{b_n, |\langle f, b_n > | \leqslant T} |\langle f, b_n \rangle|^2 + T^2 M$.

Theorem Proof

- Theshold T : Lagrangian $\mathcal{L}(f, T, \mathbf{B}) = \sum_{b_n, |\langle f, b_n > | \leqslant T} |\langle f, b_n \rangle|^2 + T^2 M$.
- **Step 1:** Existence of a bandelet basis $\mathbf{B}_0$ that satisfies

$$\mathcal{L}(f, T, \mathbf{B}) \leqslant C T^{2\alpha/(\alpha+1)}$$

  - Lemma on a single square.
  - Construction of a good partition.

Theorem Proof

- Theshold T : Lagrangian
  $\mathcal{L}(f,T,\mathbf{B}) = \sum_{b_n, |\langle f, b_n > | \leqslant T} |\langle f, b_n \rangle|^2 + T^2 M$.
- **Step 1:** Existence of a bandelet basis $\mathbf{B}_0$ that satisfies

$$\mathcal{L}(f,T,\mathbf{B}) \leqslant C T^{2\alpha/(\alpha+1)}$$

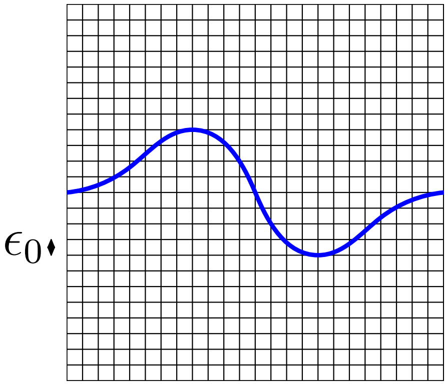
  - Lemma on a single square.
  - Construction of a good partition.
- **Step 2:** The optimization finds such a basis and it yields the right decay.

Theorem Proof

- Theshold T : Lagrangian
  $\mathcal{L}(f,T,\mathbf{B}) = \sum_{b_n, |\langle f,b_n \rangle| \leqslant T} |\langle f, b_n \rangle|^2 + T^2 M$.
- **Step 1:** Existence of a bandelet basis $\mathbf{B}_0$ that satisfies

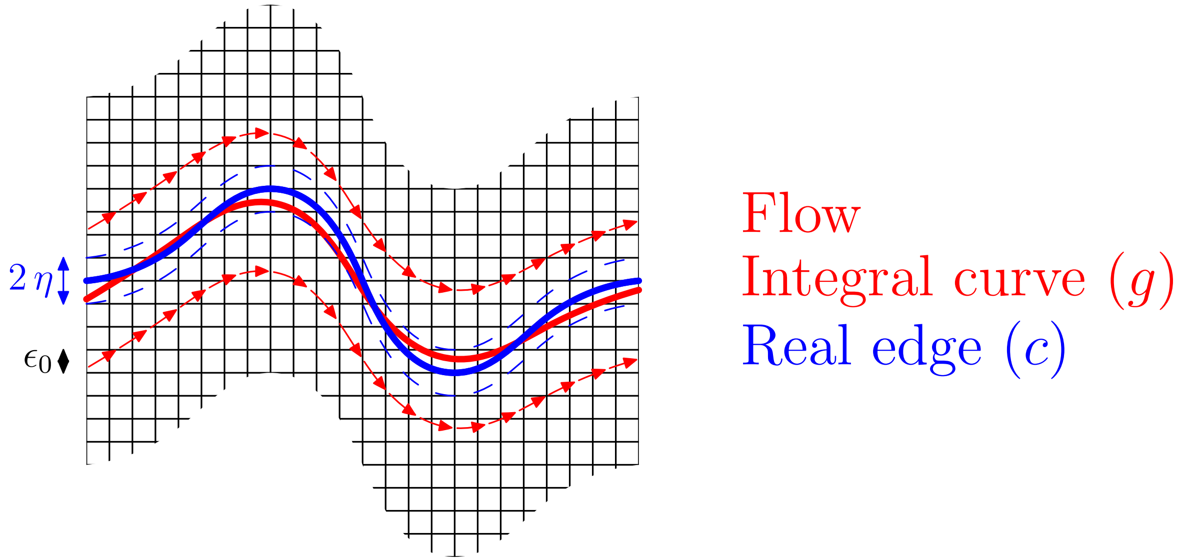
$$\mathcal{L}(f,T,\mathbf{B}) \leqslant C T^{2\alpha/(\alpha+1)}$$

  - Lemma on a single square.
  - Construction of a good partition.
- **Step 2:** The optimization finds such a basis and it yields the right decay.
- Discretization: at a resolution ϵ_0 : $T^{2\alpha/(\alpha+1)} \geqslant T^2 \geqslant \epsilon_0$.



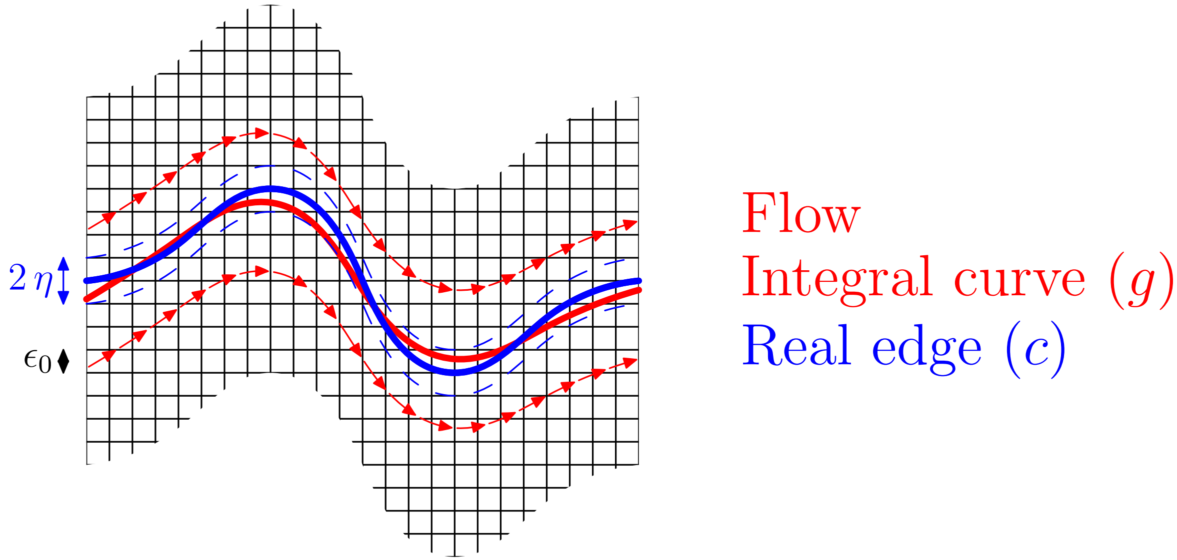
Bandelet Approximation

Bandelet Approximation



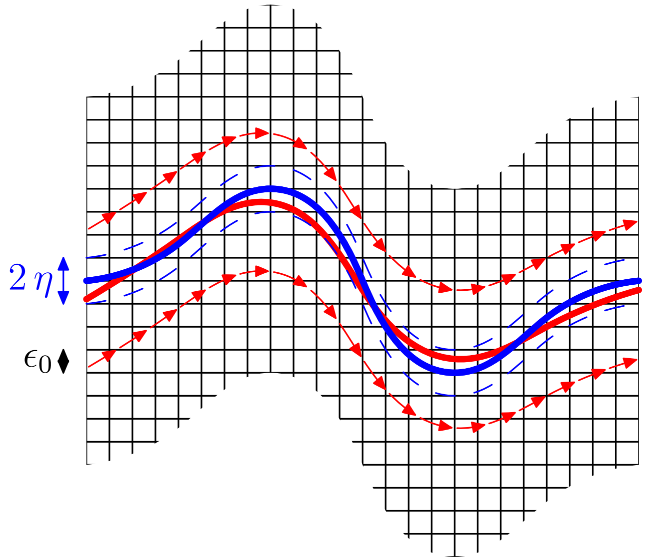
- Tube defined by g: f is warped to $Wf = f(x_1, x_2 + g(x_1))$.

Bandelet Approximation



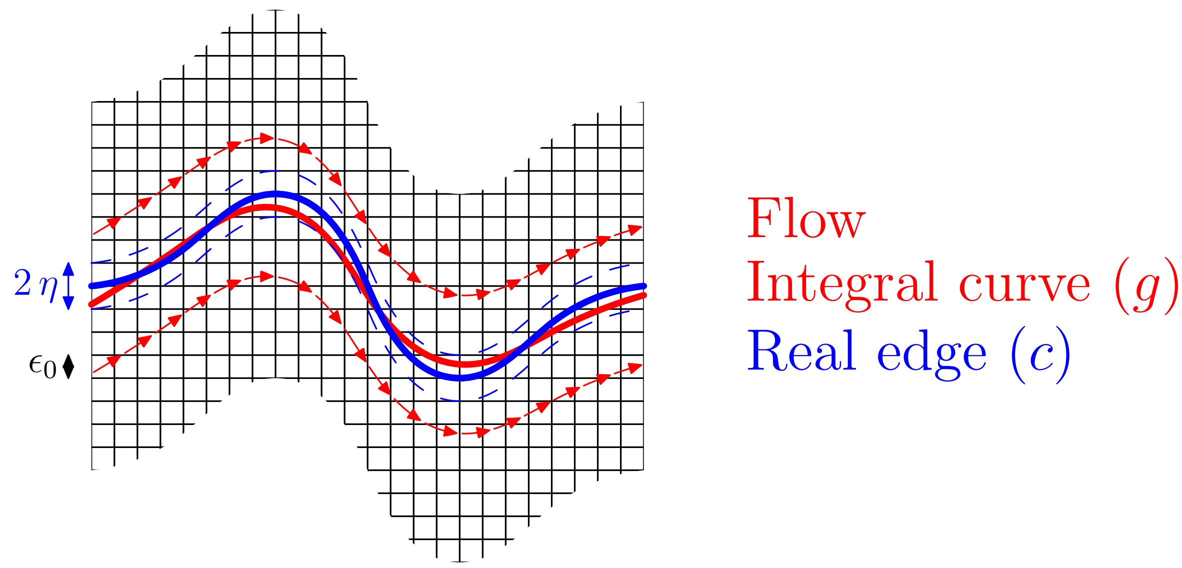
- Tube defined by g: f is warped to $Wf = f(x_1, x_2 + g(x_1))$.
- g such that
  - $|(g-c)^{(\beta)}| \leqslant C\eta^{1-\beta/\alpha}$ with $\eta = \max(s, T^{2\alpha/(\alpha+1)})$.
  - $\eta^{-1/\alpha}$ geometric parameters.

Bandelet Approximation



- Tube defined by g: f is warped to $Wf = f(x_1, x_2 + g(x_1))$.
- g such that
  - $|(g-c)^{(\beta)}| \leqslant C\eta^{1-\beta/\alpha}$ with $\eta = \max(s, T^{2\alpha/(\alpha+1)})$.
  - $\eta^{-1/\alpha}$ geometric parameters.
- $|\frac{\partial^\alpha}{\partial x_1^\alpha} Wf| \leqslant Cs^{-1}$.

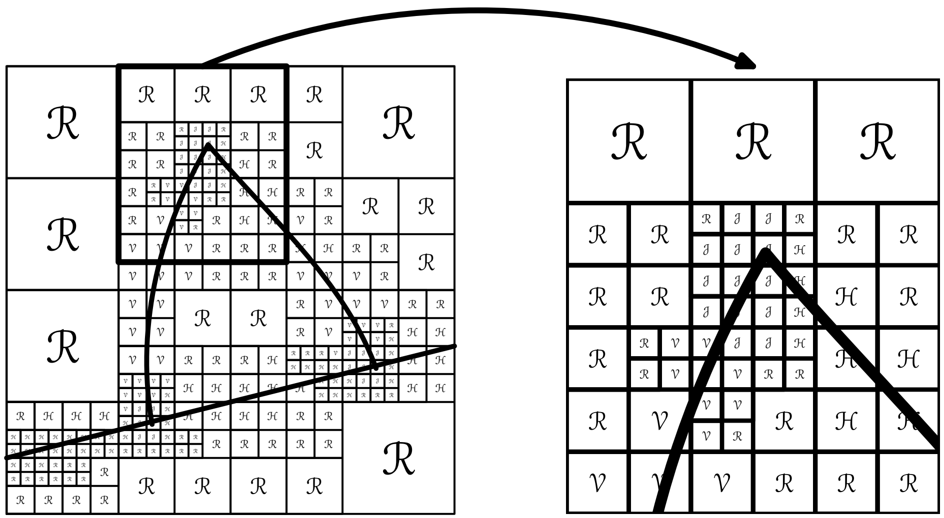
Bandelet Approximation



- Tube defined by g: f is warped to $Wf = f(x_1, x_2 + g(x_1))$.
- g such that
  - $|(g-c)^{(\beta)}| \leqslant C\eta^{1-\beta/\alpha}$ with $\eta = \max(s, T^{2\alpha/(\alpha+1)})$.
  - $\eta^{-1/\alpha}$ geometric parameters.
- $|\frac{\partial^\alpha}{\partial x_1^\alpha} Wf| \leqslant Cs^{-1}$.
- $\mathcal{L}(f, T, \mathbf{B}_i) = ||f - f_M||^2 + T^2 M \leqslant CT^{2\alpha/(\alpha+1)}$.

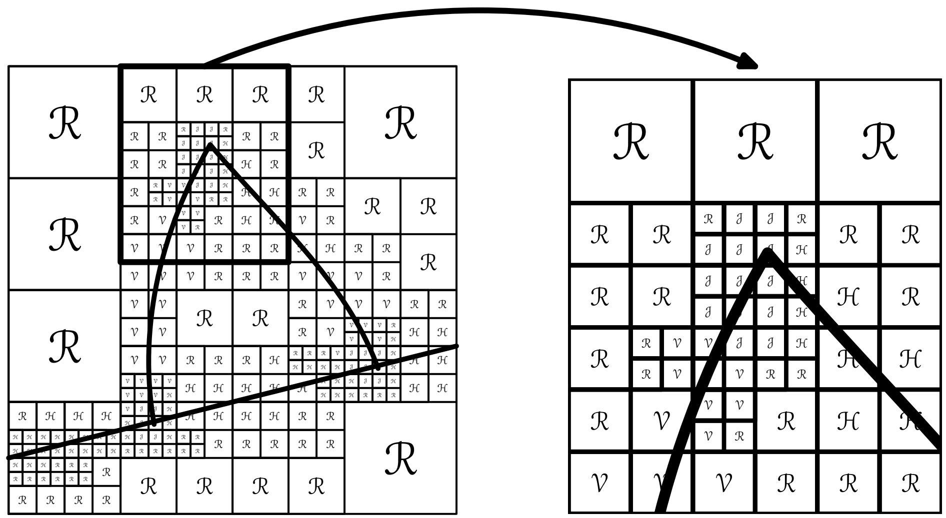
Partition

Partition



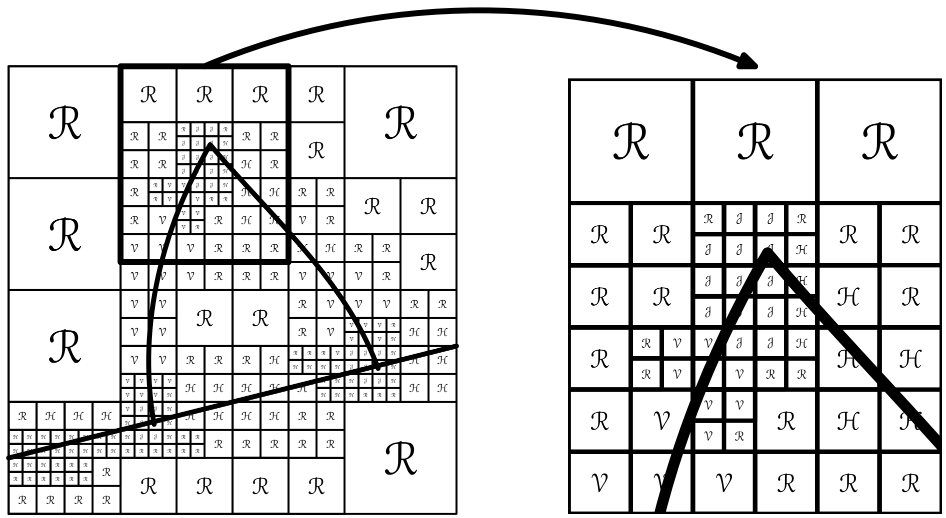
- Quadtree to obtain a partition with:
  - at most $K(1 + |\log(\eta)|)$ edge squares with one single horizontal or vertical *edge* (bandelets),
  - a finite number of junctions square of size η (wavelets),
  - at most $K(1 + |\log(\eta)|)$ regular squares (wavelets).

Partition



- Quadtree to obtain a partition with:
  - at most $K(1+|\log(\eta)|)$ edge squares with one single horizontal or vertical *edge* (bandelets),
  - a finite number of junctions square of size η (wavelets),
  - at most $K(1+|\log(\eta)|)$ regular squares (wavelets).
- Total geometric cost : $C\eta^{-1/\alpha}$.

Partition



- Quadtree to obtain a partition with:
  - at most $K(1+|\log(\eta)|)$ edge squares with one single horizontal or vertical *edge* (bandelets),
  - a finite number of junctions square of size η (wavelets),
  - at most $K(1+|\log(\eta)|)$ regular squares (wavelets).
- Total geometric cost : $C\eta^{-1/\alpha}$.
- $$\mathcal{L}(f,T,\mathbf{B}) = \sum_i \mathcal{L}(f,T,\mathbf{B}_i) \leqslant CT^{2\alpha/(\alpha+1)}$$

Optimization

Optimization

- Existence of a partition (previous construction) such that the Lagrangian satisfies

$$\mathcal{L}(f, T, \mathbf{B}_0) \leqslant C\, T^{2\alpha/(\alpha+1)} \quad .$$

Optimization

- Existence of a partition (previous construction) such that the Lagrangian satisfies

$$\mathcal{L}(f, T, \mathbf{B}_0) \leqslant C\, T^{2\alpha/(\alpha+1)} \quad .$$

- CART algorithm minimizes $\mathcal{L}(f, T, \mathbf{B})$ and tests this partition.

Optimization

- Existence of a partition (previous construction) such that the Lagrangian satisfies

$$\mathcal{L}(f, T, \mathbf{B}_0) \leqslant C\, T^{2\alpha/(\alpha+1)} \quad .$$

- CART algorithm minimizes $\mathcal{L}(f, T, \mathbf{B})$ and tests this partition.
- The optimal function f_M satisfies thus also

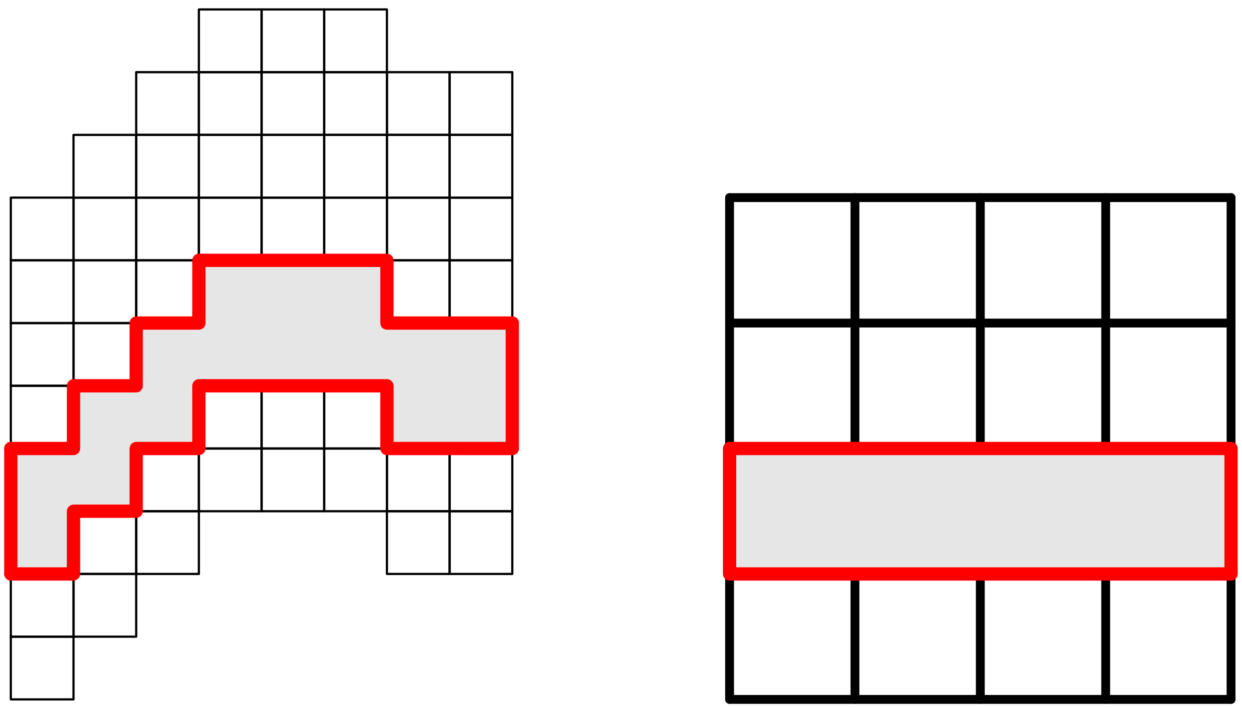
$$\|f - f_M\|^2 + T^2 M \leqslant \mathcal{L}(f, T, \mathbf{B}_0) \leqslant C\, T^{2\alpha/(\alpha+1)}$$

and so

$$\|f - f_M\|^2 \leqslant K M^{-1/\alpha} \quad .$$

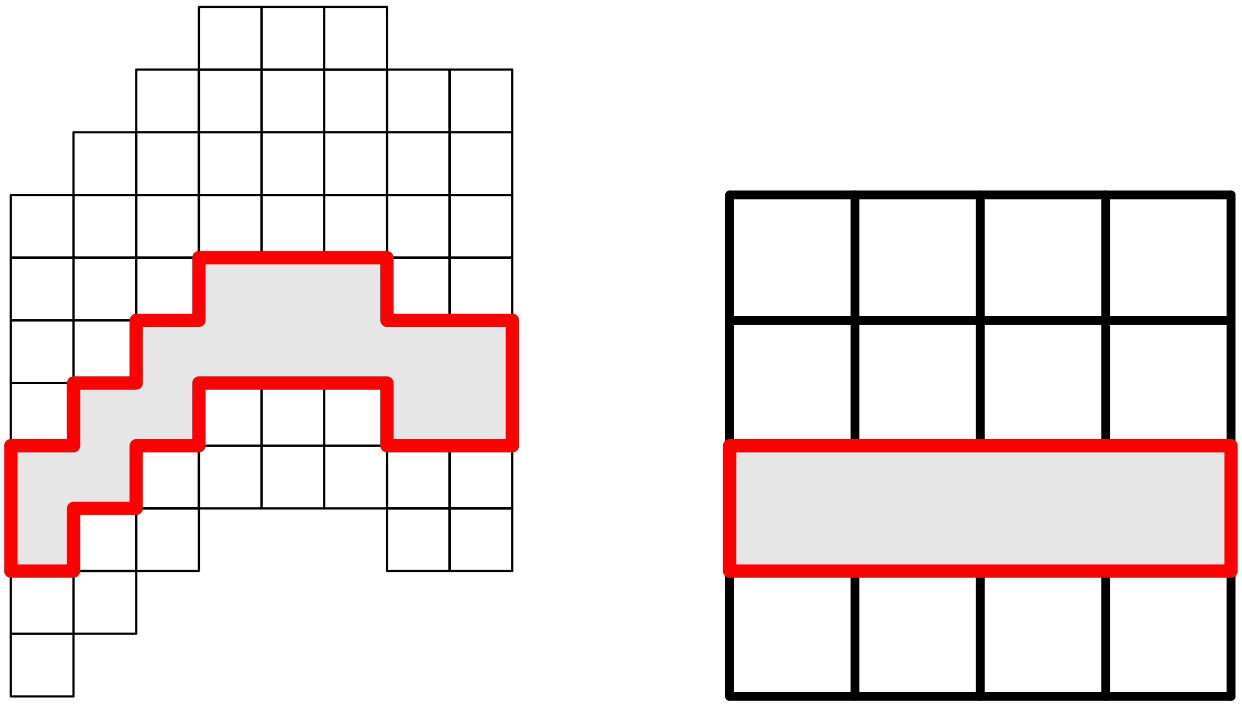
Discrete Setting

Discrete Setting



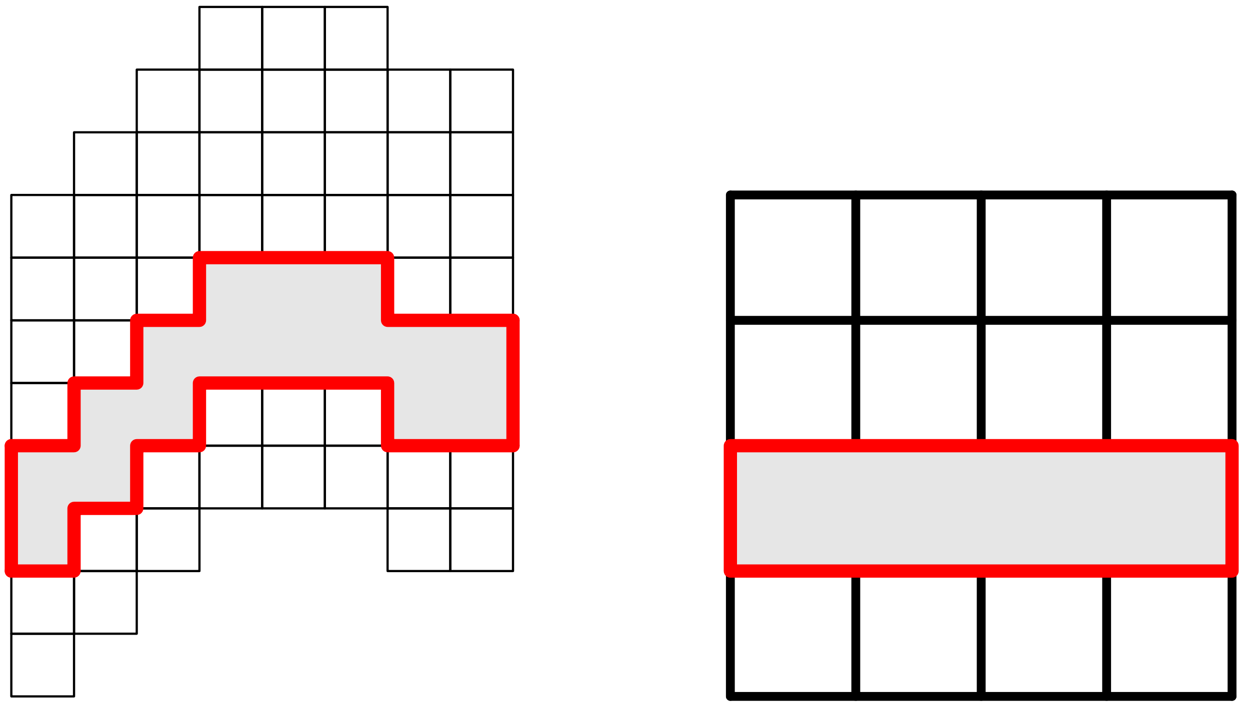
- Interpolation of order 0 (no interpolation) leads to discrete bandelets.

Discrete Setting



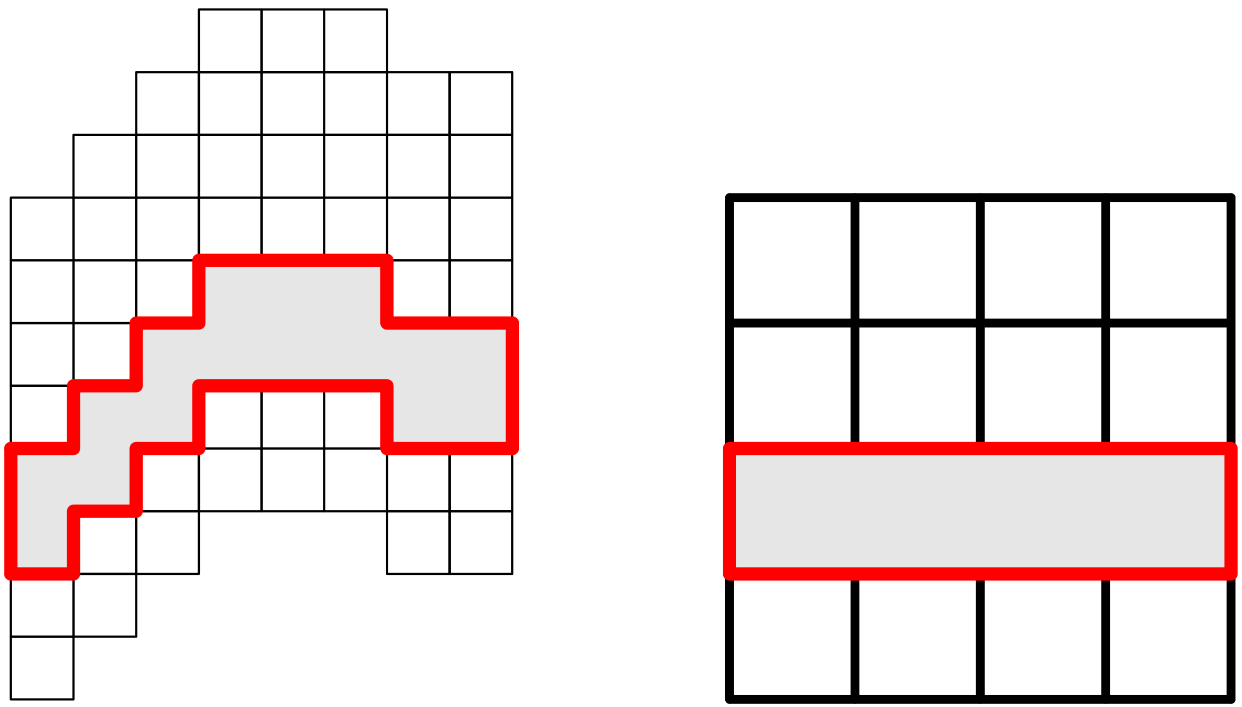
- Interpolation of order 0 (no interpolation) leads to discrete bandelets.
- Discrete basis/frame.

Discrete Setting



- Interpolation of order 0 (no interpolation) leads to discrete bandelets.
- Discrete basis/frame.
- Same optimization scheme.

Discrete Setting



- Interpolation of order 0 (no interpolation) leads to discrete bandelets.
- Discrete basis/frame.
- Same optimization scheme.
- Same kind of theoretical results.

Naive Coder

Naive Coder

- Approximation scheme leads to compression scheme.

Naive Coder

- Approximation scheme leads to compression scheme.
- Simple prototype coder:
  - Quad-Tree coded with 1 bits per inner node.
  - Geometry coded with $\log_2 3$ bits for the type of bandelets, $\log_2 T$ bits for the geometry scale and $\log_2 T$ bits per geometric coefficients.
  - Position of the M largest coefficients: $(1 - \log(M/N))$ bits per coefficients.
  - Value of the M largest coefficients: $\log_2 T$ bits per coefficients.

Naive Coder

- Approximation scheme leads to compression scheme.
- Simple prototype coder:
  - Quad-Tree coded with 1 bits per inner node.
  - Geometry coded with $\log_2 3$ bits for the type of bandelets, $\log_2 T$ bits for the geometry scale and $\log_2 T$ bits per geometric coefficients.
  - Position of the M largest coefficients: $(1 - \log(M/N))$ bits per coefficients.
  - Value of the M largest coefficients: $\log_2 T$ bits per coefficients.
- Sufficient to use the previous theorem to obtain:

$$D(R) \leqslant C\, R^{-\alpha}\, (\log R)^{\alpha}$$

Naive Coder

- Approximation scheme leads to compression scheme.
- Simple prototype coder:
  - Quad-Tree coded with 1 bits per inner node.
  - Geometry coded with $\log_2 3$ bits for the type of bandelets, $\log_2 T$ bits for the geometry scale and $\log_2 T$ bits per geometric coefficients.
  - Position of the M largest coefficients: $(1 - \log(M/N))$ bits per coefficients.
  - Value of the M largest coefficients: $\log_2 T$ bits per coefficients.
- Sufficient to use the previous theorem to obtain:

$$D(R) \leqslant C\, R^{-\alpha}\, (\log R)^{\alpha}$$

- No claim of optimality for the coder.

Conclusion

Conclusion

- Bandelets algorithmic closely related to wavelet algorithmic.

Conclusion

- Bandelets algorithmic closely related to wavelet algorithmic.
- Optimization can lead to computationnal issue.

Conclusion

- Bandelets algorithmic closely related to wavelet algorithmic.
- Optimization can lead to computationnal issue.
- Non linear approximation theorems.

Conclusion

- Bandelets algorithmic closely related to wavelet algorithmic.
- Optimization can lead to computationnal issue.
- Non linear approximation theorems.
- Some differences between theory and practice.

Overview

- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. Dossal)
  - Denoising
  - Deconvolution of seismic data
- Session 4
  - Bandelets NG

Denoising

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

Denoising

- Estimate an image f from the noisy data

  $$X = f + W \quad \text{where } W \text{ is Gaussian white of variance } \sigma^2.$$

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

  $$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \; .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.

Denoising

- Estimate an image f from the noisy data

  $X = f + W$ where W is Gaussian white of variance σ^2.

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

$$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \; .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.
- The *oracle model* minimizes the risk $\mathsf{E}\left(\|F - f\|^2\right)$.

Denoising

- Estimate an image f from the noisy data

  $$X = f + W \quad \text{where } W \text{ is Gaussian white of variance } \sigma^2.$$

- Thresholding estimator in a basis $\mathbf{B} = \{g_m\}_{1 \leqslant m \leqslant N^2}$:

  $$F = \sum_{|\langle X, g_m \rangle| > T} \langle X, g_m \rangle \, g_m = P_{\mathcal{M}}(X) \ .$$

- Model: subspace $\mathcal{M}$ of a bandelet frame associated to a geometry.
- The *oracle model* minimizes the risk $\mathsf{E}\left(\|F - f\|^2\right)$.
- Design of a penalized estimator:

  $$\|F - X\|^2 + \lambda \, \mathsf{Pen}(F)$$

MDL Approach

MDL Approach

- MDL = Bayesian.

MDL Approach

- MDL $=$ Bayesian.
- Prior on the functions: $P(f) \propto 2^{-R}$ where R is the number of bits required to code the function.

MDL Approach

- MDL $=$ Bayesian.
- Prior on the functions: $P(f) \propto 2^{-R}$ where R is the number of bits required to code the function.
- Estimator selects the model that minimizes

$$\|X - F\|^2 + \lambda \, \sigma^2 \, R$$

MDL Approach

- MDL $=$ Bayesian.
- Prior on the functions: $P(f) \propto 2^{-R}$ where R is the number of bits required to code the function.
- Estimator selects the model that minimizes

$$\|X - F\|^2 + \lambda\,\sigma^2\,R$$

- Allows to reuse the compression algorithm almost directly.

MDL Approach

- MDL = Bayesian.
- Prior on the functions: $P(f) \propto 2^{-R}$ where R is the number of bits required to code the function.
- Estimator selects the model that minimizes

$$\|X - F\|^2 + \lambda \, \sigma^2 \, R$$

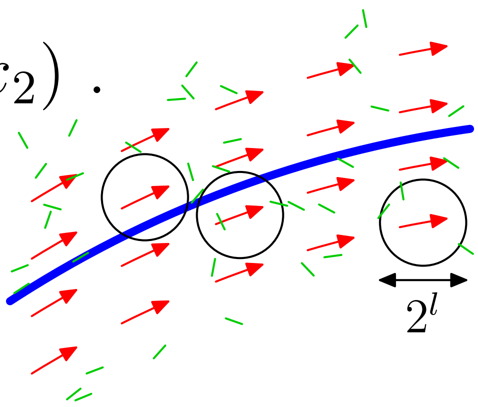
- Allows to reuse the compression algorithm almost directly.
- No theoretical results but a practical algorithm with a flow estimation.

Computation of Flow with Noise

Computation of Flow with Noise

- Image regularization at a scale 2^l: $f \star \theta_l$ with

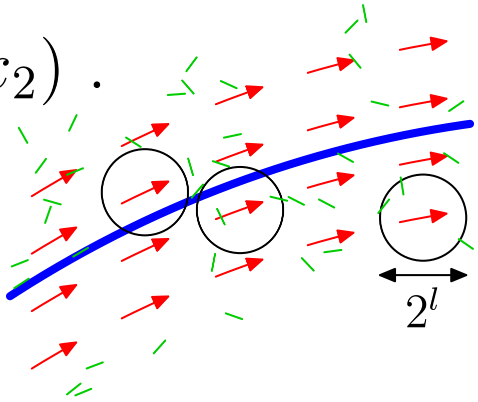
$$\theta_l(x_1, x_2) = 2^{-2l}\,\theta(2^{-l}x_1\,,\,2^{-l}x_2)\ .$$



Computation of Flow with Noise

- Image regularization at a scale 2^l: $f \star \theta_l$ with

$$\theta_l(x_1, x_2) = 2^{-2l}\, \theta(2^{-l} x_1\,,\, 2^{-l} x_2)\ .$$



- Flow parameterized at the scale 2^l:

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n\, \phi(2^{-l} x - n)\ ,$$

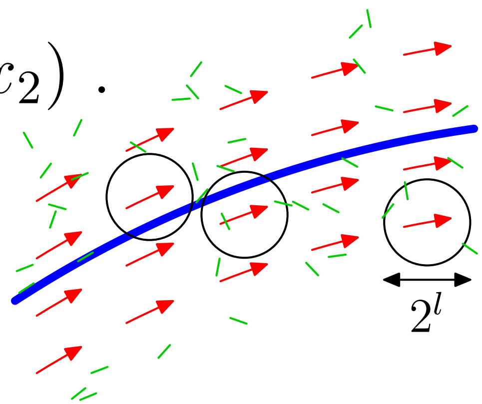
where the α_n minimize

$$\int_\Omega \left| \nabla (f \star \theta_l)(x_1, x_2)\, .\, \vec{\tau}(x_1, x_2) \right|^2 dx_1 dx_2 = \int_\Omega \left| \frac{\partial (f \star \theta_l)(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 dx_1 dx_2$$

Computation of Flow with Noise

- Image regularization at a scale 2^l: $f \star \theta_l$ with

$$\theta_l(x_1, x_2) = 2^{-2l}\, \theta(2^{-l} x_1\, ,\, 2^{-l} x_2)\ .$$



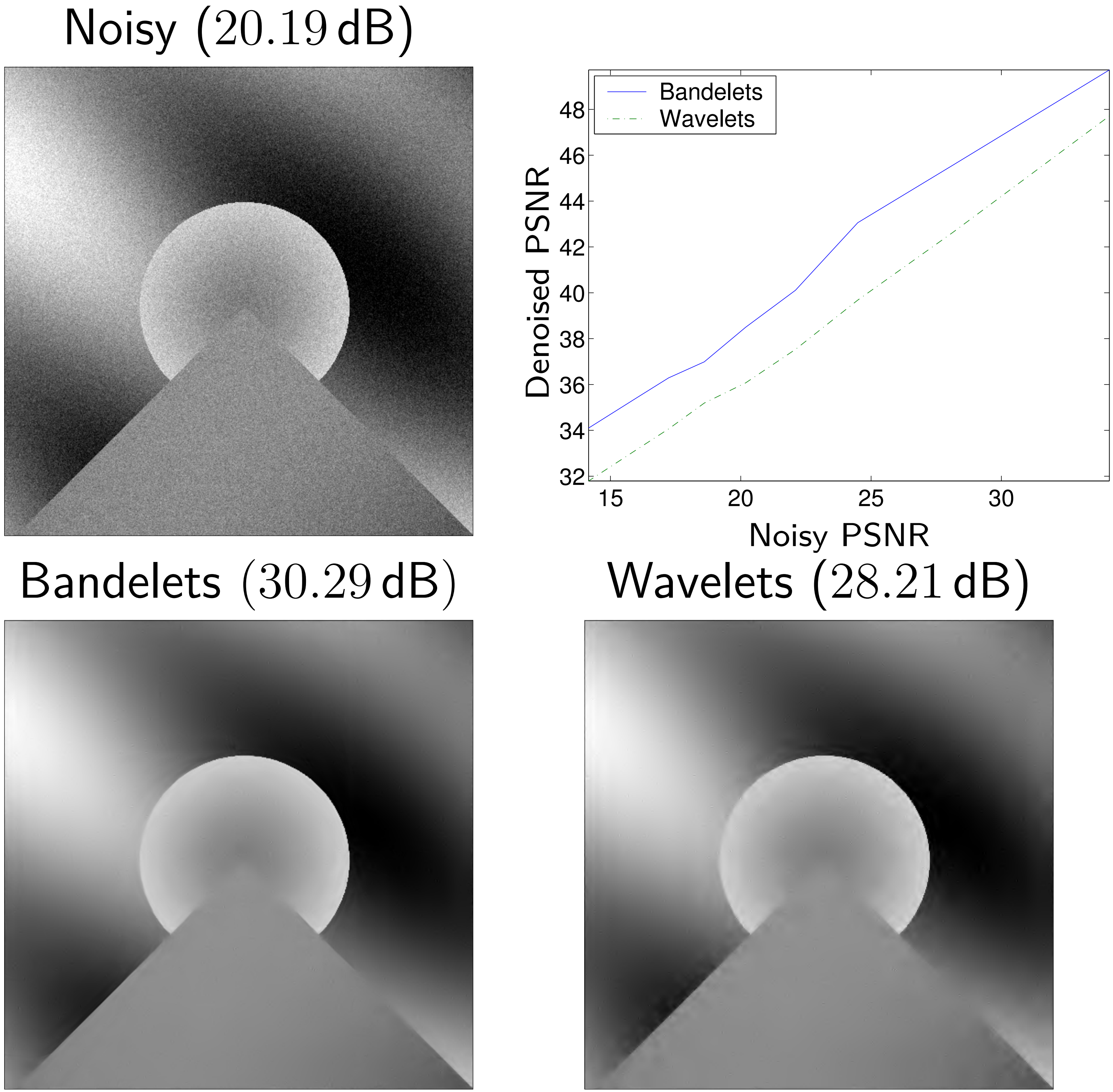
- Flow parameterized at the scale 2^l:

$$c'(x) = \sum_{n=1}^{L2^{-l}} \alpha_n\, \phi(2^{-l} x - n)\ ,$$

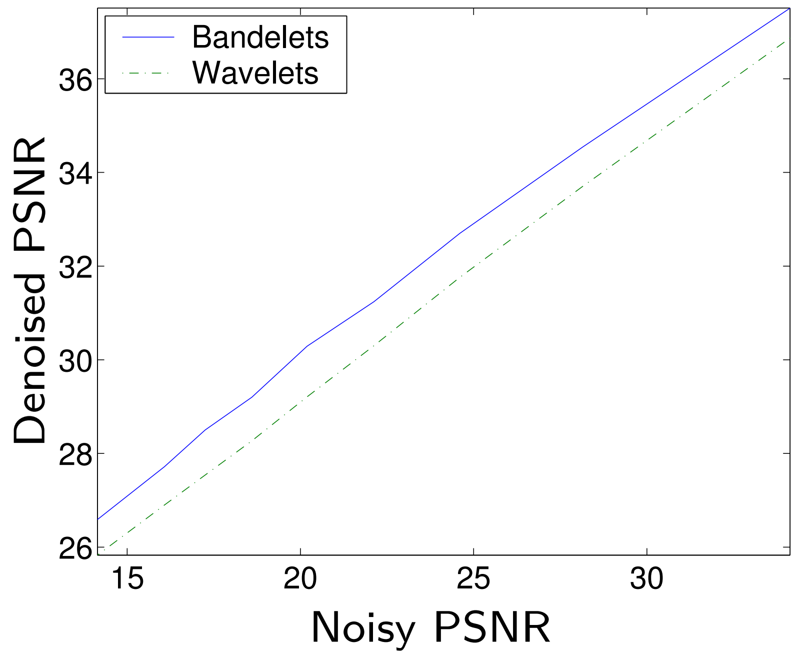
where the α_n minimize

$$\int_\Omega \left| \nabla (f \star \theta_l)(x_1, x_2)\, .\, \vec{\tau}(x_1, x_2) \right|^2 dx_1 dx_2 = \int_\Omega \left| \frac{\partial (f \star \theta_l)(x_1, x_2)}{\partial \vec{\tau}(x_1, x_2)} \right|^2 dx_1 dx_2$$

- The scale 2^l is computed by the penalized minimization. It is adjusted to the noise variance and the local geometric signal regularity.

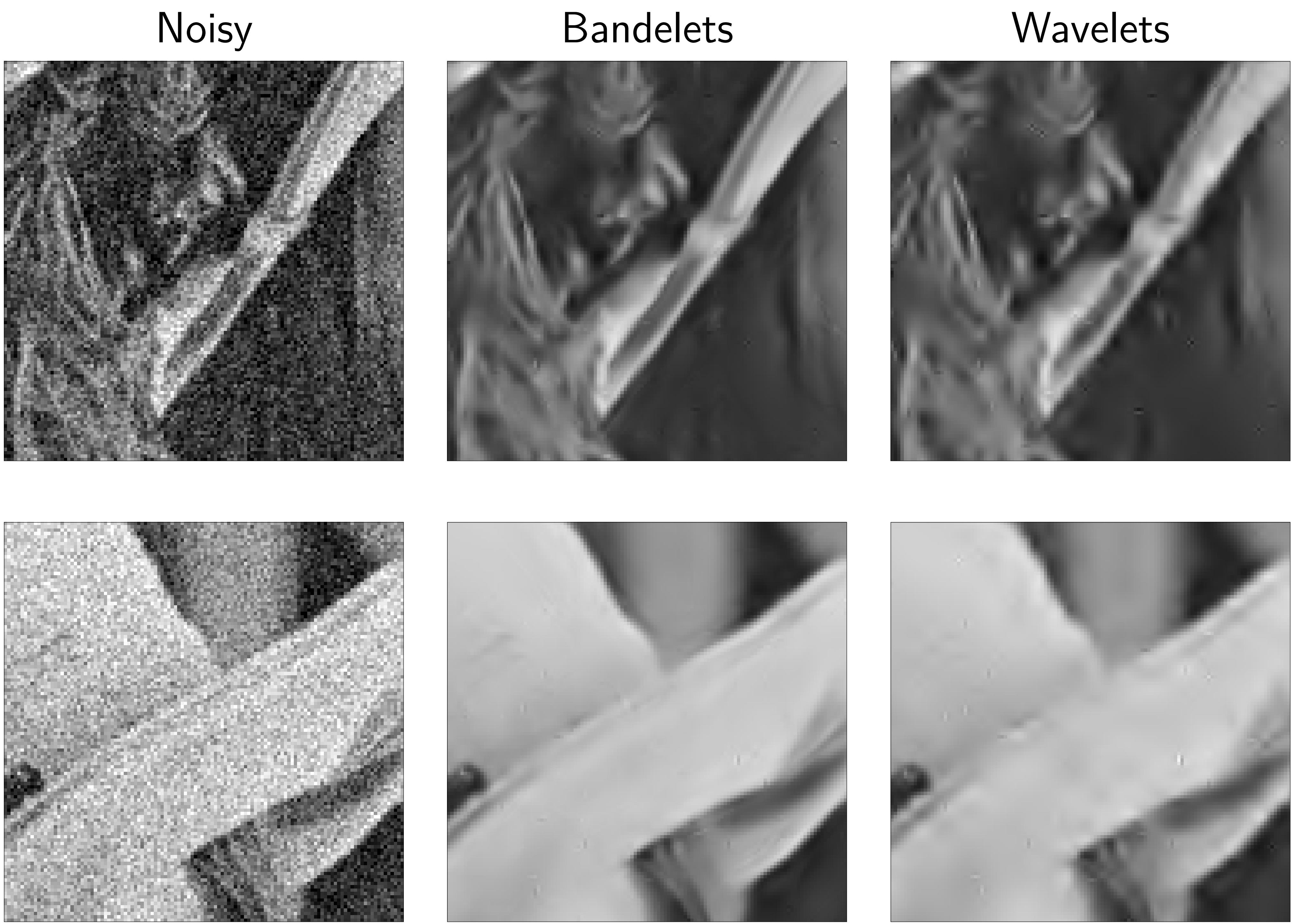

Noisy (20.19 dB)
Bandelets
Wavelets
Denoised PSNR
48
46
44
42
40
38
36
34
32
15
20
25
30
Noisy PSNR
Bandelets (30.29 dB)
Wavelets (28.21 dB)

Noisy $(20.19\,\text{dB})$

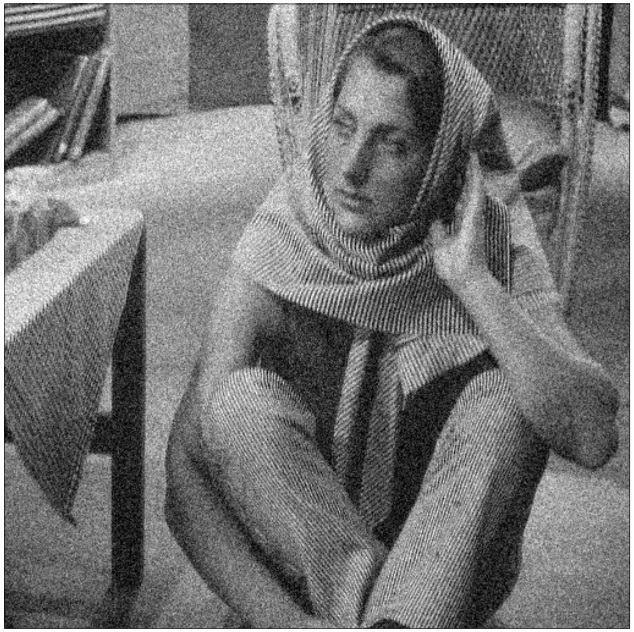


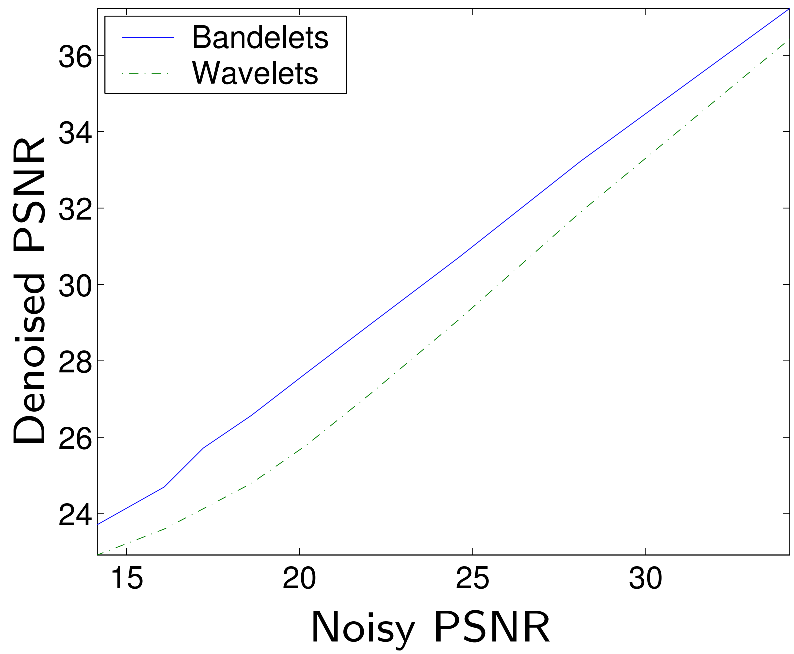
Bandelets $(30.29\,\text{dB})$

Wavelets $(28.21\,\text{dB})$


Noisy
Bandelets
Wavelets

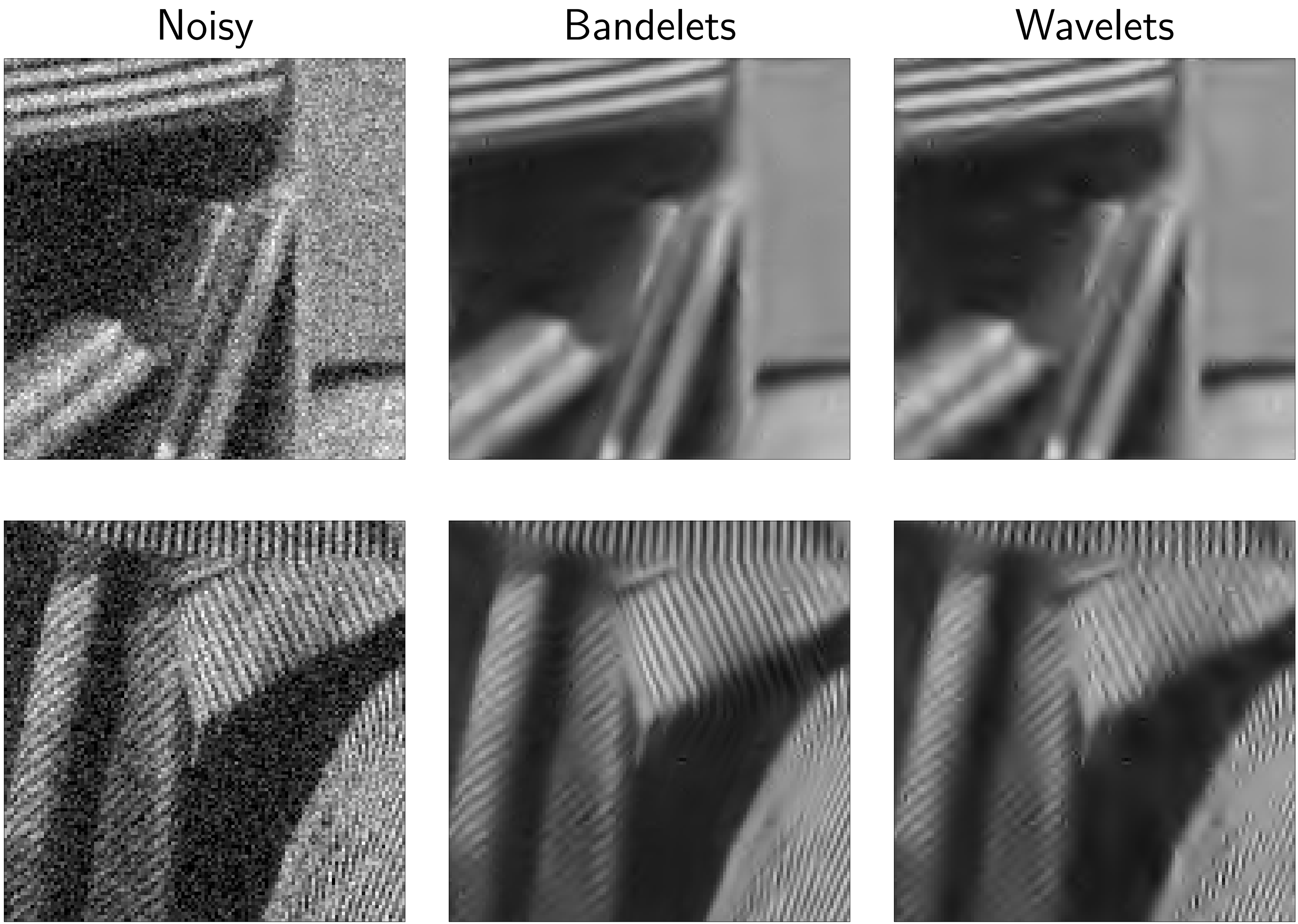
Noisy ($20.19\,\mathrm{dB}$)





Bandelets ($27.68\,\mathrm{dB}$)

Wavelets ($25.79\,\mathrm{dB}$)


Noisy
Bandelets
Wavelets

Penalized model selection

Penalized model selection

- How to obtain a theoretical result of convergence for a bandelet estimator (Ch. *Dossal*)?

Penalized model selection

- How to obtain a theoretical result of convergence for a bandelet estimator (Ch. *Dossal*)?
- Control on the total number ν of bandelets in all the different tested models.

Penalized model selection

- How to obtain a theoretical result of convergence for a bandelet estimator (Ch. *Dossal*)?
- Control on the total number ν of bandelets in all the different tested models.
- Minimizing of

$$-\|F\|^2 + \lambda\,\sigma^2\,(\log \nu)\,M$$

gives an almost optimal result on the estimator risk (*Donoho*).

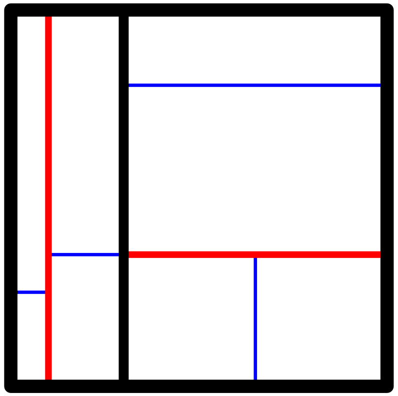
Penalized model selection

- How to obtain a theoretical result of convergence for a bandelet estimator (Ch. *Dossal*)?
- Control on the total number ν of bandelets in all the different tested models.
- Minimizing of
$$-\|F\|^2 + \lambda\,\sigma^2\,(\log\nu)\,M$$
gives an almost optimal result on the estimator risk (*Donoho*).
- But requires a orthogonal basis or a tight frame.

Orthonormal basis of bandelets

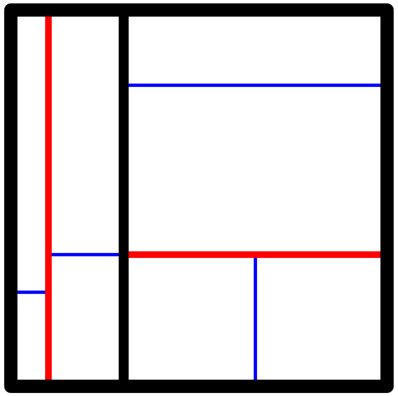
Orthonormal basis of bandelets

Orthonormal basis of bandelets



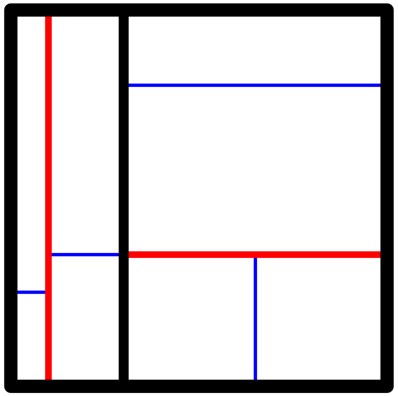
- Modifications of the border bandelets in a rectangle to obtain an orthonormal basis with suitable approximation properties.
- Splitting in rectangle with a polynomial flow.

Orthonormal basis of bandelets



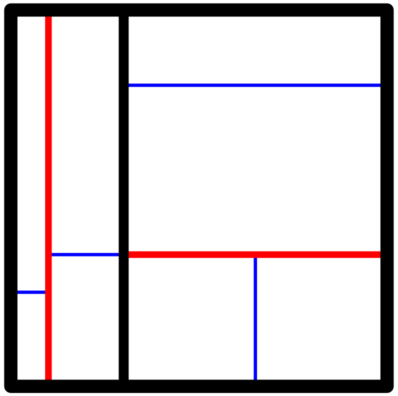
- Modifications of the border bandelets in a rectangle to obtain an orthonormal basis with suitable approximation properties.
- Splitting in rectangle with a polynomial flow.
- Tree structure preserved.

Orthonormal basis of bandelets



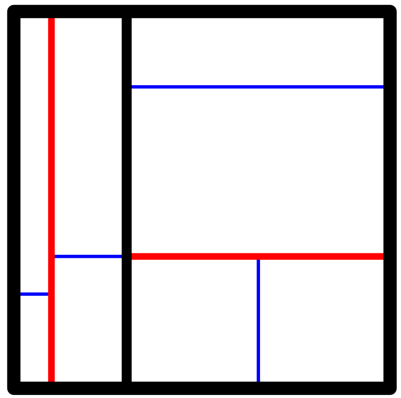
- Modifications of the border bandelets in a rectangle to obtain an orthonormal basis with suitable approximation properties.
- Splitting in rectangle with a polynomial flow.
- Tree structure preserved.
- Polynomial total number of bandelets: $\nu \leqslant C\, N^{(p+5)}$.

Orthonormal basis of bandelets



- Modifications of the border bandelets in a rectangle to obtain an orthonormal basis with suitable approximation properties.
- Splitting in rectangle with a polynomial flow.
- Tree structure preserved.
- Polynomial total number of bandelets: $\nu \leqslant C\, N^{(p+5)}$.
- Optimization complexity still polynomial.

Orthonormal basis of bandelets



- Modifications of the border bandelets in a rectangle to obtain an orthonormal basis with suitable approximation properties.
- Splitting in rectangle with a polynomial flow.
- Tree structure preserved.
- Polynomial total number of bandelets: $\nu \leqslant C\, N^{(p+5)}$.
- Optimization complexity still polynomial.
- Near optimality of the risk for the minimization of

$$-\|F\|^2 + \lambda\,\sigma^2\,(\log \nu)\, M$$

with $\log \nu \propto \log N$.

Theoretical Result

Theoretical Result

- Estimate F obtained from $Y = f + W$, where W is a white noise of variance σ^2, by minimizing
  $-\|F\|^2 + \lambda\,\sigma^2\,(\log\nu)\,M \quad .$

Theoretical Result

- Estimate F obtained from $Y = f + W$, where W is a white noise of variance σ^2, by minimizing $-\|F\|^2 + \lambda\,\sigma^2\,(\log\nu)\,M$.
- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$ with some non tangency conditions) then the estimate F satisfies

$$\|f - F\|^2 \leqslant C(\log\nu)^{\alpha/(\alpha+1)}(\log\log\nu)^{1/(\alpha+1)}\sigma^{2\alpha/(\alpha+1)}$$

with a probability greater than $1 - 2\nu^{-1/4}$.

Theoretical Result

- Estimate F obtained from $Y = f + W$, where W is a white noise of variance σ^2, by minimizing $-\|F\|^2 + \lambda \sigma^2 (\log \nu) M$.
- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$ with some non tangency conditions) then the estimate F satisfies

  $$\|f - F\|^2 \leqslant C(\log \nu)^{\alpha/(\alpha+1)} (\log \log \nu)^{1/(\alpha+1)} \sigma^{2\alpha/(\alpha+1)}$$

  with a probability greater than $1 - 2\nu^{-1/4}$.
- Unknown degree of smoothness α.

Theoretical Result

- Estimate F obtained from $Y = f + W$, where W is a white noise of variance σ^2, by minimizing $-\|F\|^2 + \lambda\,\sigma^2\,(\log\nu)\,M$.
- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$ with some non tangency conditions) then the estimate F satisfies

$$\|f - F\|^2 \leqslant C(\log\nu)^{\alpha/(\alpha+1)}(\log\log\nu)^{1/(\alpha+1)}\sigma^{2\alpha/(\alpha+1)}$$

with a probability greater than $1 - 2\nu^{-1/4}$.

- Unknown degree of smoothness α.
- Near optimal decay exponent α.

Theoretical Result

- Estimate F obtained from $Y = f + W$, where W is a white noise of variance σ^2, by minimizing $-\|F\|^2 + \lambda\sigma^2 (\log\nu)\, M$.
- **Theorem:** If f is $\mathbf{C}^\alpha$ geometrically regular ($f = \tilde{f}$ or $f = \tilde{f} \star h$ with $\tilde{f}$ $\mathbf{C}^\alpha$ outside a set of curves, that are by parts $\mathbf{C}^\alpha$ with some non tangency conditions) then the estimate F satisfies
$$\|f - F\|^2 \leqslant C(\log\nu)^{\alpha/(\alpha+1)}(\log\log\nu)^{1/(\alpha+1)}\sigma^{2\alpha/(\alpha+1)}$$
with a probability greater than $1 - 2\nu^{-1/4}$.
- Unknown degree of smoothness α.
- Near optimal decay exponent α.
- Sampling case: $Y(x_i) = f(x_i) + \epsilon(x_i)$ with ϵ a white noise of variance σ^2:
$$\|f - F\|^2 \leqslant C\left(\frac{\log N}{N^2}\right)^{\alpha/(\alpha+1)}(\log\log N)^{1/(\alpha+1)}$$

Risk Minimization

Risk Minimization

- Risk : $E(\|f - F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.

Risk Minimization

- Risk : $E(\|f - F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.
- Oracle risk : minimum risk over all possible estimator ($\mathcal{M} = \{n$ s.t. $|\langle f, g_n \rangle| \geqslant \sigma\}$ in the best basis).

Risk Minimization

- Risk : $E(\|f - F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.
- Oracle risk : minimum risk over all possible estimator ($\mathcal{M} = \{n$ s.t. $|\langle f, g_n \rangle| \geqslant \sigma\}$ in the best basis).
- Oracle estimate: $f_0 = \text{argmin}_g \|f - g\|^2 + \sigma^2 M(g)$

Risk Minimization

- Risk : $E(\|f - F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.
- Oracle risk : minimum risk over all possible estimator ($\mathcal{M} = \{n$ s.t. $|\langle f, g_n \rangle| \geqslant \sigma\}$ in the best basis).
- Oracle estimate: $f_0 = \text{argmin}_g \|f - g\|^2 + \sigma^2 M(g)$
- Oracle estimate v2: $f_1 = \text{argmin}_g \|f - g\|^2 + \lambda\sigma^2(\log \nu) M(g)$

Risk Minimization

- Risk : $E(\|f - F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.
- Oracle risk : minimum risk over all possible estimator ($\mathcal{M} = \{n$ s.t. $|\langle f, g_n \rangle| \geqslant \sigma\}$ in the best basis).
- Oracle estimate: $f_0 = \text{argmin}_g \|f - g\|^2 + \sigma^2 M(g)$
- Oracle estimate v2: $f_1 = \text{argmin}_g \|f - g\|^2 + \lambda\sigma^2(\log \nu)M(g)$
- Empirical estimate: $F = \text{argmin}_g \|X - g\|^2 + \lambda\sigma^2(\log \nu)M(g)$

Risk Minimization

- Risk : $E(\|f-F\|^2) = \sum_{n \notin \mathcal{M}} |\langle f, g_n \rangle|^2 + M\sigma^2$.
- Oracle risk : minimum risk over all possible estimator ($\mathcal{M} = \{n$ s.t. $|\langle f, g_n \rangle| \geqslant \sigma\}$ in the best basis).
- Oracle estimate: $f_0 = \text{argmin}_g \|f-g\|^2 + \sigma^2 M(g)$
- Oracle estimate v2: $f_1 = \text{argmin}_g \|f-g\|^2 + \lambda\sigma^2(\log\nu)M(g)$
- Empirical estimate: $F = \text{argmin}_g \|X-g\|^2 + \lambda\sigma^2(\log\nu)M(g)$
- Theorem: With high probability

$$\|f-F\|^2 + \lambda\sigma^2(\log\nu)M(F) \leqslant C\|f-f_1\|^2 + \lambda\sigma^2(\log\nu)M(f_1)$$

$$\|f-F\|^2 + \lambda\sigma^2(\log\nu)M(F) \leqslant C(\log\nu)(\|f-f_0\|^2 + \sigma^2 M(f_0))$$

Proof – 1

Proof – 1

- *Donoho*: quiet easy (once we assume the following lemma).

Proof – 1

- *Donoho*: quiet easy (once we assume the following lemma).
- **Lemma:** With high probability, $\forall \mathcal{M}, ||P_{\mathcal{M}} W||^2 \leqslant 4M\sigma^2$.

Proof – 1

- *Donoho*: quiet easy (once we assume the following lemma).
- **Lemma:** With high probability, $\forall \mathcal{M}, ||P_{\mathcal{M}} W||^2 \leqslant 4M\sigma^2$.
- Step 1:

$$\|f - f_1\|^2 + \lambda\sigma^2(\log\nu)M(f_1) \leqslant \lambda(\log\nu)(\|f - f_0\|^2 + \sigma^2 M(f_0))$$

(easy).

Proof – 1

- *Donoho*: quiet easy (once we assume the following lemma).
- **Lemma:** With high probability, $\forall \mathcal{M}, ||P_{\mathcal{M}} W||^2 \leqslant 4M\sigma^2$.
- Step 1:

$$\|f - f_1\|^2 + \lambda\sigma^2(\log \nu)M(f_1) \leqslant \lambda(\log \nu)(\|f - f_0\|^2 + \sigma^2 M(f_0))$$

(easy).

- Step 2:

$$\|f - F\|^2 + \lambda\sigma^2(\log \nu)M(F) \leqslant C\|f - f_1\|^2 + \lambda\sigma^2(\log \nu)M(f_1).$$

Proof – 2

- $\|X-g\|^2 = \|X-f\| + 2\langle X-f, f-g\rangle + \|f-g\|^2$.
- Inserting this in
  $\|X-F\|^2 + \lambda\sigma^2(\log\nu)M(F) \leqslant \|X-f_1\|^2 + \lambda\sigma^2(\log\nu)M(f_1)$
  yields

$$\|f-F\|^2 + \lambda\sigma^2(\log\nu)M(F) \leqslant \|f-f_1\|^2 + \lambda\sigma^2(\log\nu)M(f_1) + 2\langle X-f, F-f_1\rangle \quad .$$

- Now $\langle X-f, F-f_1\rangle \leqslant \|P_{\mathcal{M}\cup\mathcal{M}_1}W\|\|F-f_1\|$.
- By definition of f_1,

$$\|F-f_1\| \leqslant \|F-f\| + \|f-f_1\| \leqslant 2(\|f-F\|^2 + \lambda\sigma^2(\log\nu)M(F))^{1/2} \quad .$$

- With the lemma,

$$\|P_{\mathcal{M}\cup\mathcal{M}_1}W\|^2 \leqslant 4\sigma^2\log\nu(M(F)+M(f_1))$$
$$\|P_{\mathcal{M}\cup\mathcal{M}_1}W\|^2 \leqslant (8/\lambda)\sigma^2(\|f-F\|^2 + \lambda\sigma^2(\log\nu)M(F))$$

- Combining this two last bounds gives the result.

Conclusion

Conclusion

- Bandelets : nice example of model selection.

Conclusion

- Bandelets : nice example of model selection.
- Both practical and theoretical results.

Conclusion

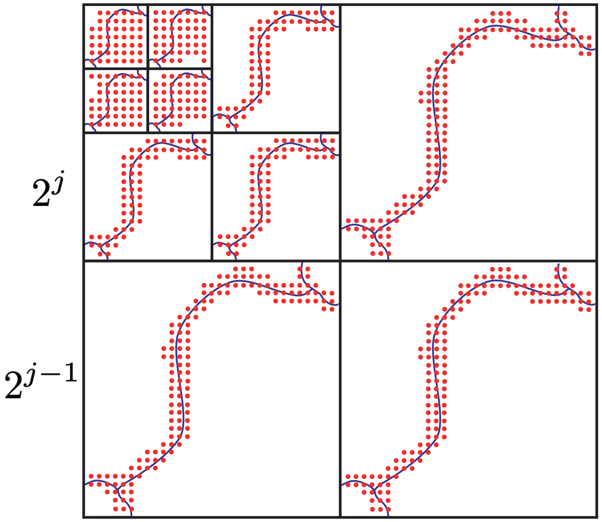
- Bandelets : nice example of model selection.
- Both practical and theoretical results.
- Bandelets are well adapted to seismic data deconvolution.

Overview

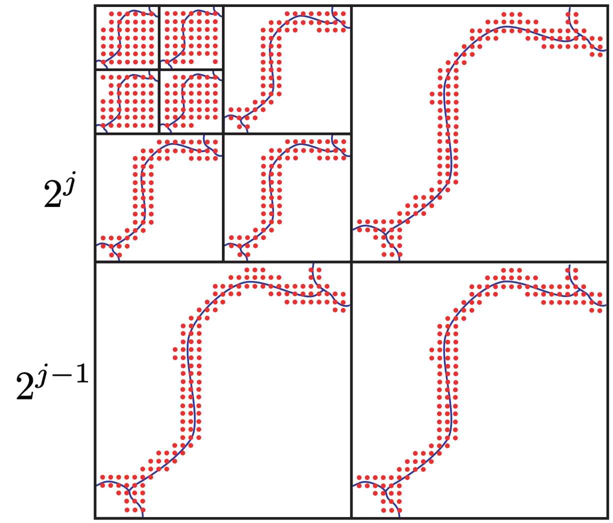
- Session 1
  - Bandelets construction
  - Non linear approximation with bandelets
  - Compression
- Session 2
  - Bandelets algorithmic
  - Non linear approximation theorem(s)
- Session 3 (with Ch. Dossal)
  - Denoising
  - Deconvolution of seismic data
- Session 4
  - Bandelets NG

Return to Wavelet Coefficients

Return to Wavelet Coefficients

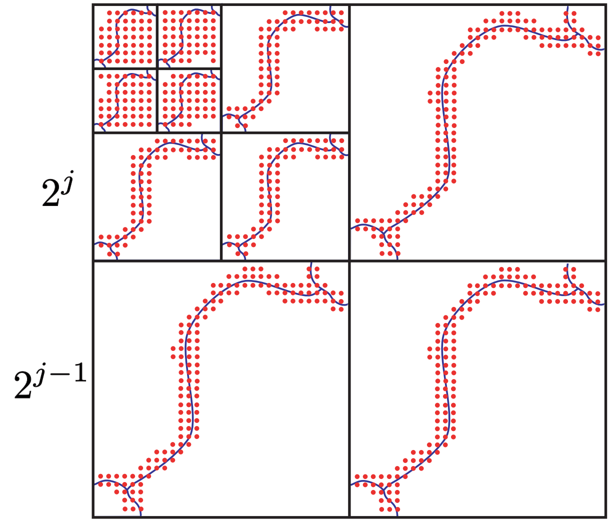


Return to Wavelet Coefficients



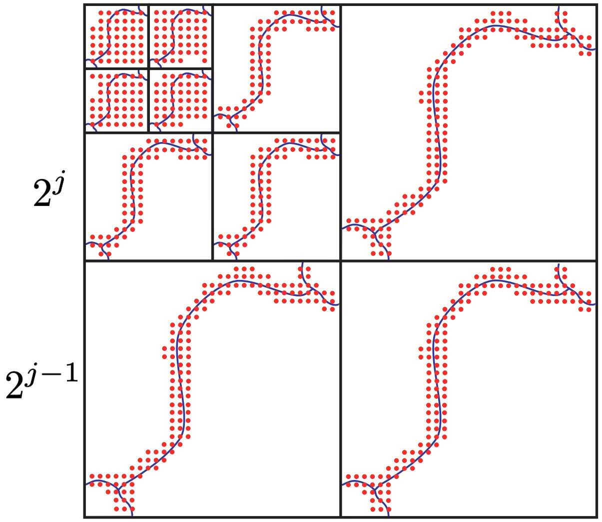
- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?

Return to Wavelet Coefficients



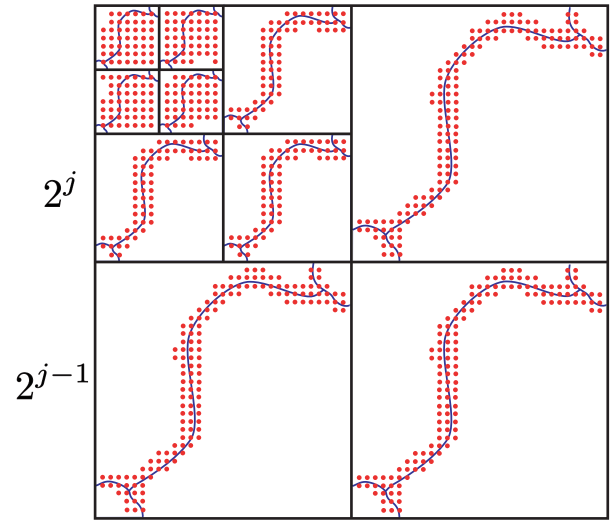
- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.

Return to Wavelet Coefficients



- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.

Return to Wavelet Coefficients

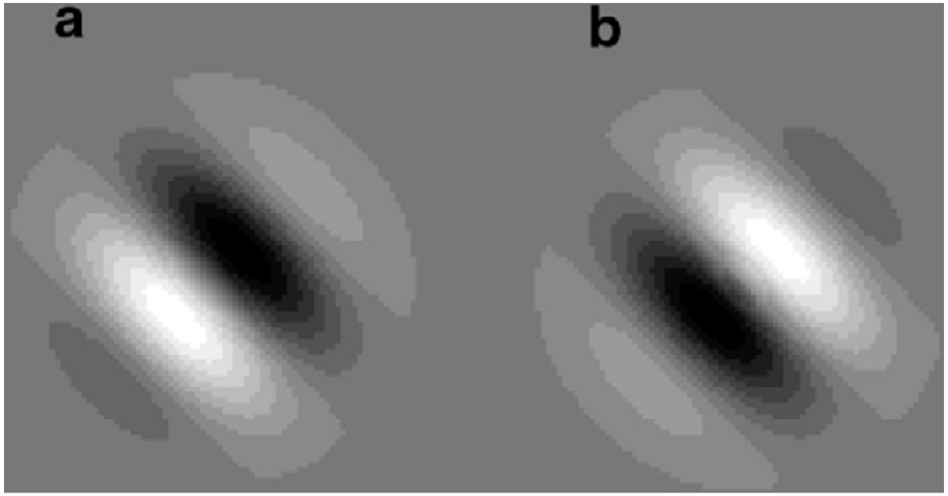


- At each scale, how to approximate the vector of non-zero wavelet coefficients (chaotic behavior) ?
- Use of parameterized models projected over wavelets: "wedgelets" and "wedgeprints" by *Baraniuk, Romberg, Wakin* and *Dragotti, Vetterli*: discontinuities along parameterized curves.
- Difficult to parameterize smooth edges $f = \tilde{f} \star \theta_s$.
- Modification of the wavelet transform (*Cohen*).

Geometry in the Visual Brain

Geometry in the Visual Brain

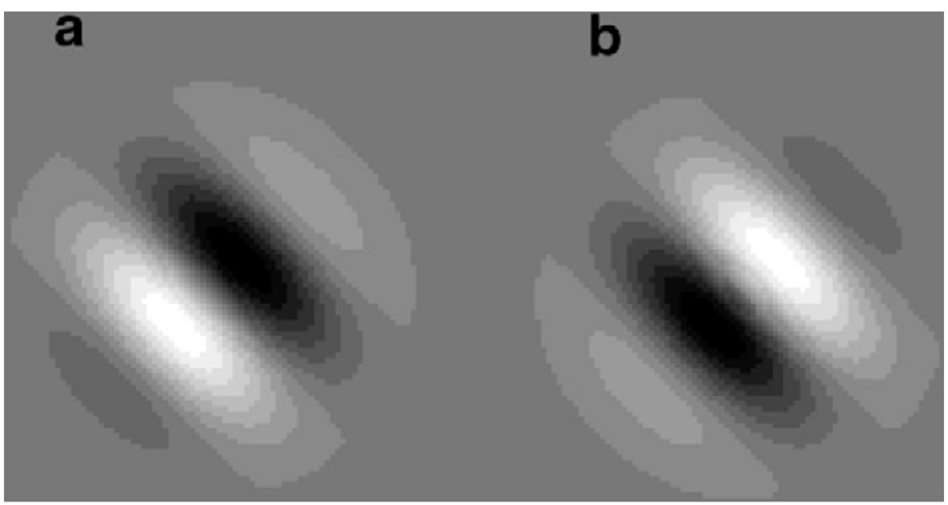
- Simple cells in V1 provide inner products with wavelets:



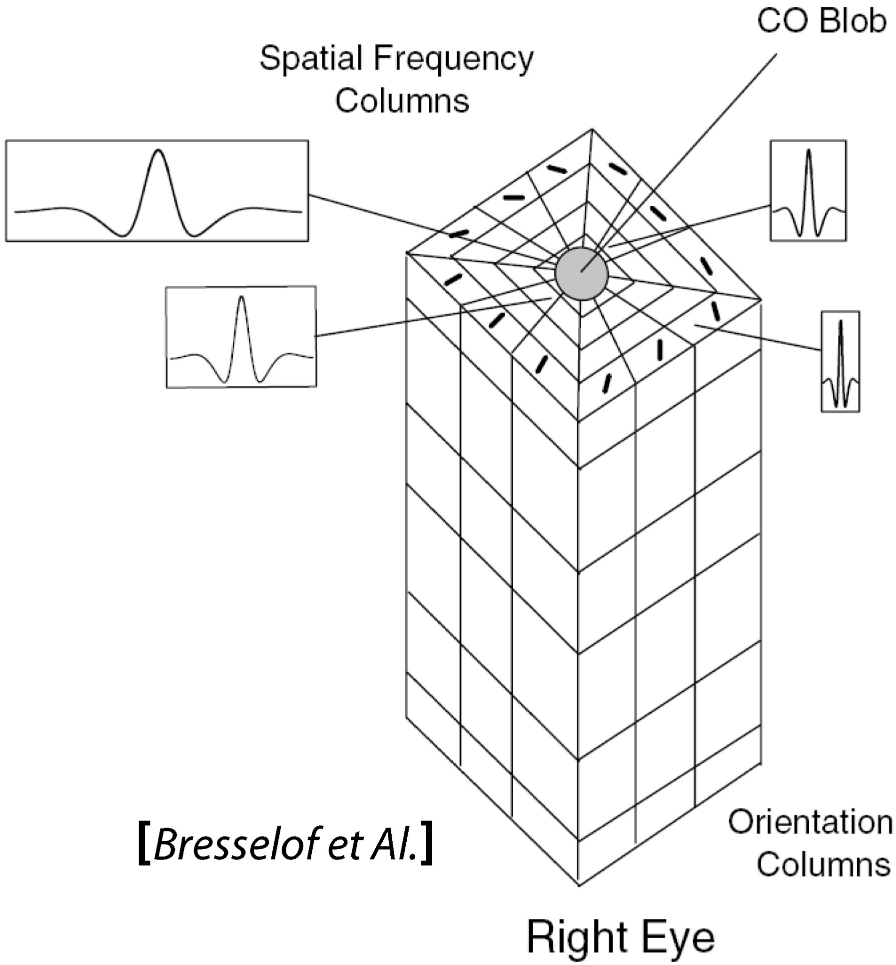
[*Wolf et al*]

Geometry in the Visual Brain

- Simple cells in V1 provide inner products with wavelets:



[*Wolf et al*]

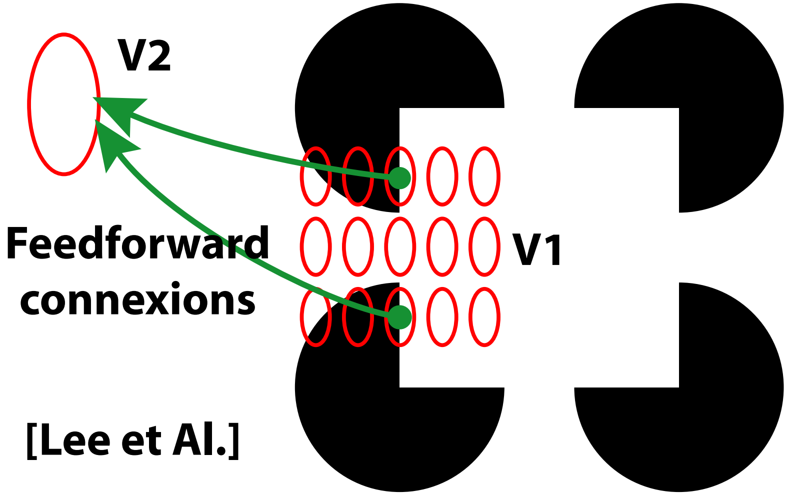


[*Bresselof et Al.*]

Contour Integration in Physiology

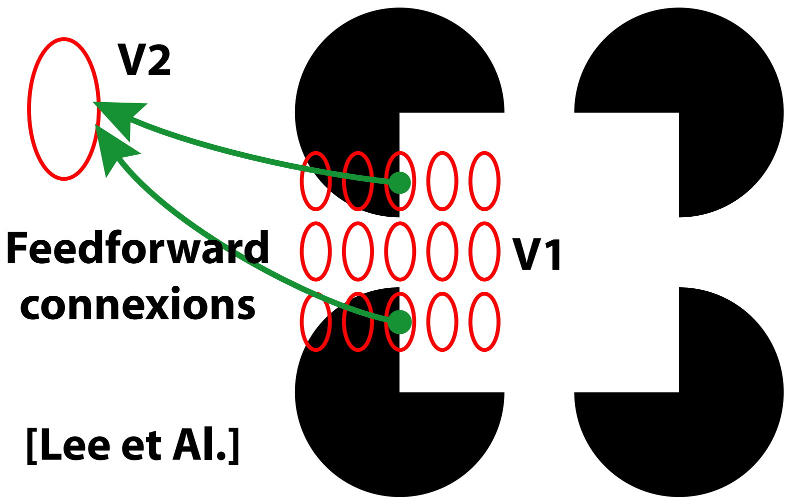
Contour Integration in Physiology

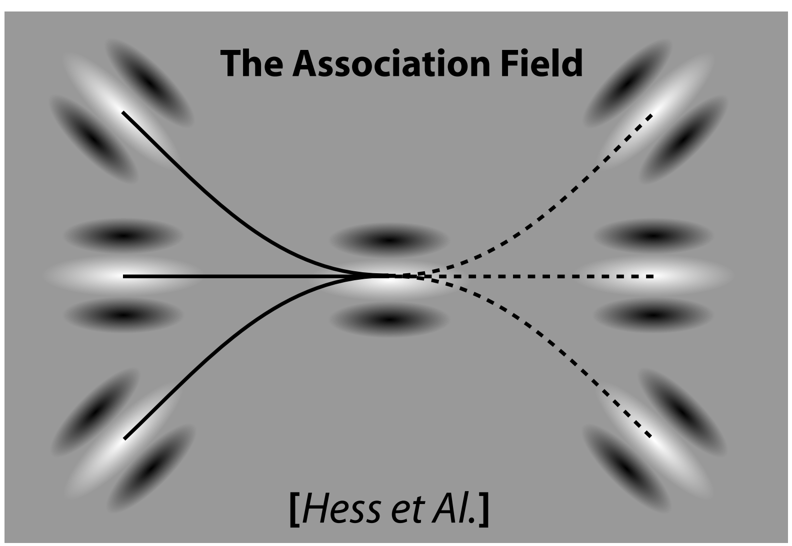
- Contour integration in V2:



Contour Integration in Physiology

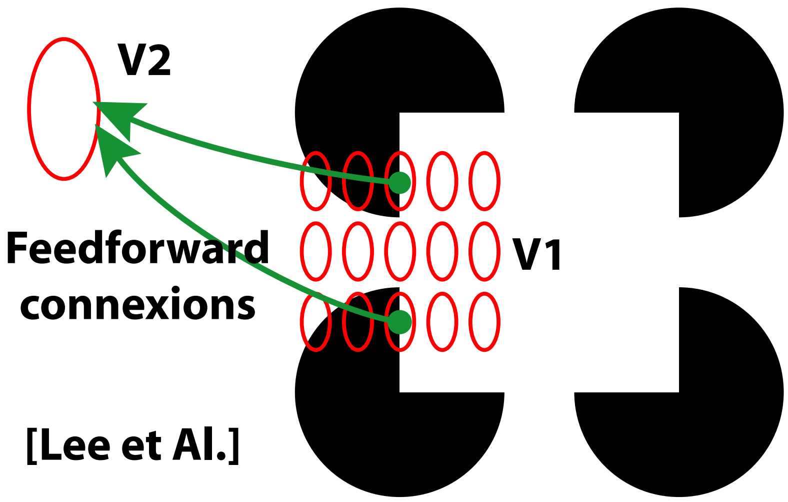
- Contour integration in V2:

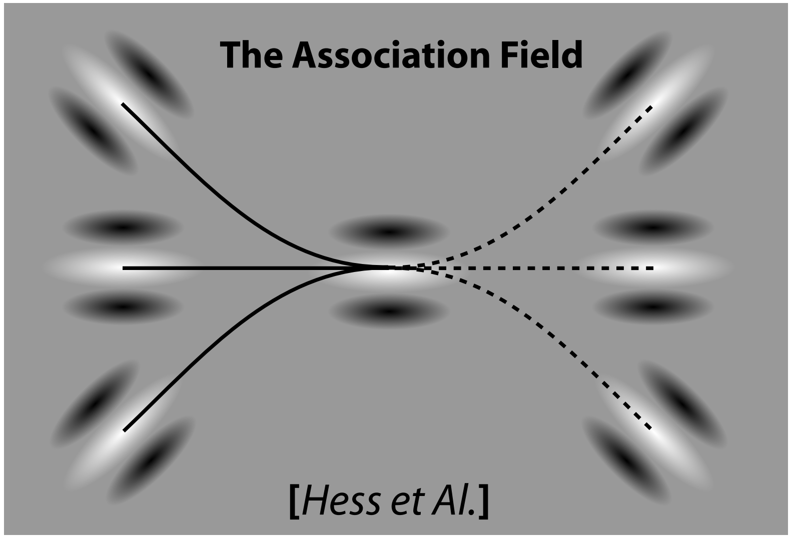


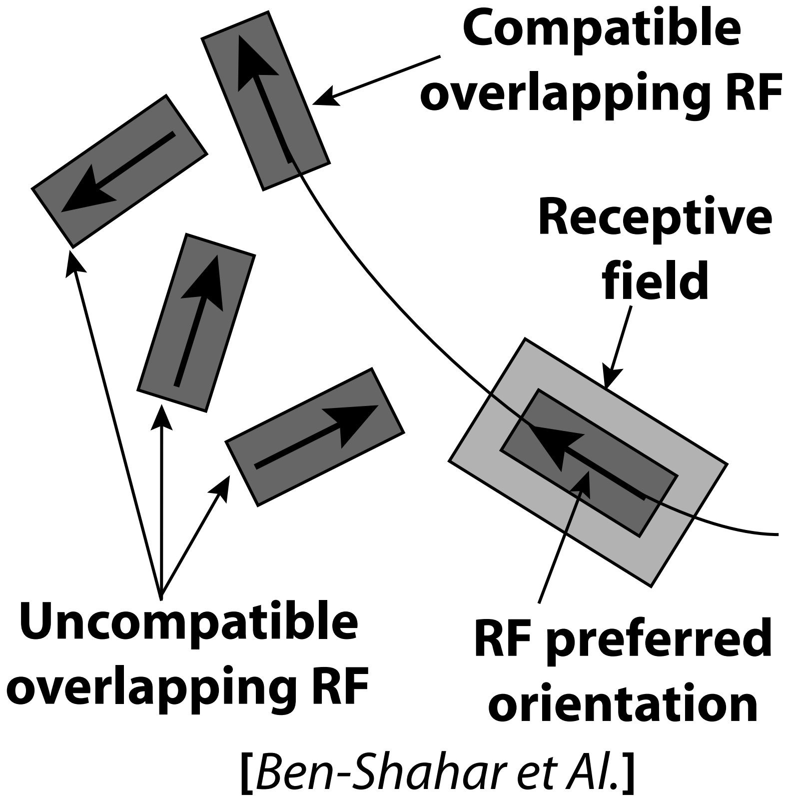


Contour Integration in Physiology

- Contour integration in V2:







Warping the Wavelet Space

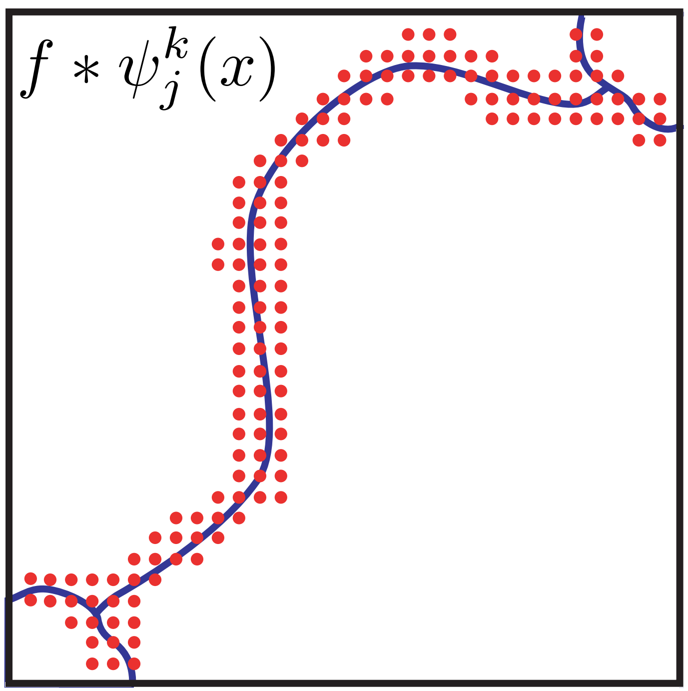
Gabriel Peyré

Warping the Wavelet Space

Gabriel Peyré

- Wavelet coefficients are samples of a regularized function:

$$\langle f\,,\,\psi_{j,n}^k\rangle = f \star \psi_j^k(2^j n) \quad \text{with} \quad \psi_j^k(x) = 2^{-j}\,\psi^k(-2^{-j}x)\quad .$$

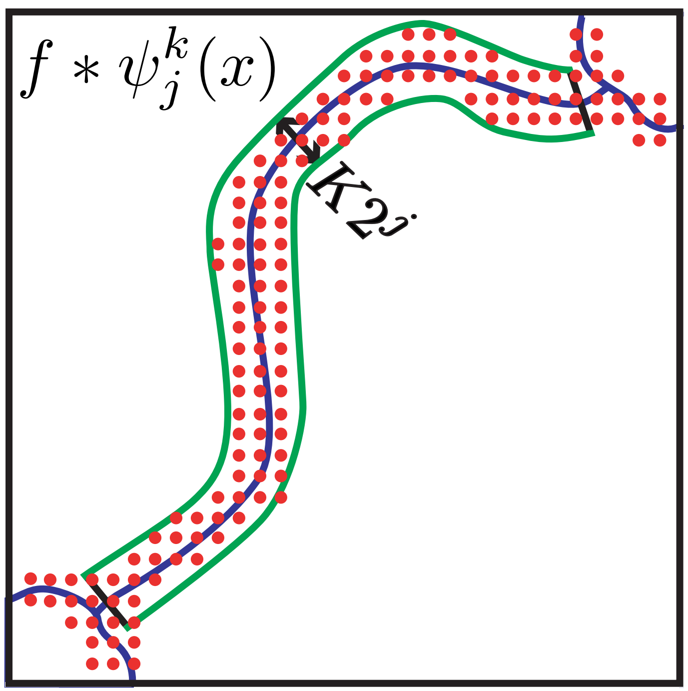


Warping the Wavelet Space

Gabriel Peyré

- Wavelet coefficients are samples of a regularized function:

$$\langle f\,,\,\psi_{j,n}^k\rangle = f \star \psi_j^k(2^j n) \quad \text{with} \quad \psi_j^k(x) = 2^{-j}\,\psi^k(-2^{-j}x) \quad .$$

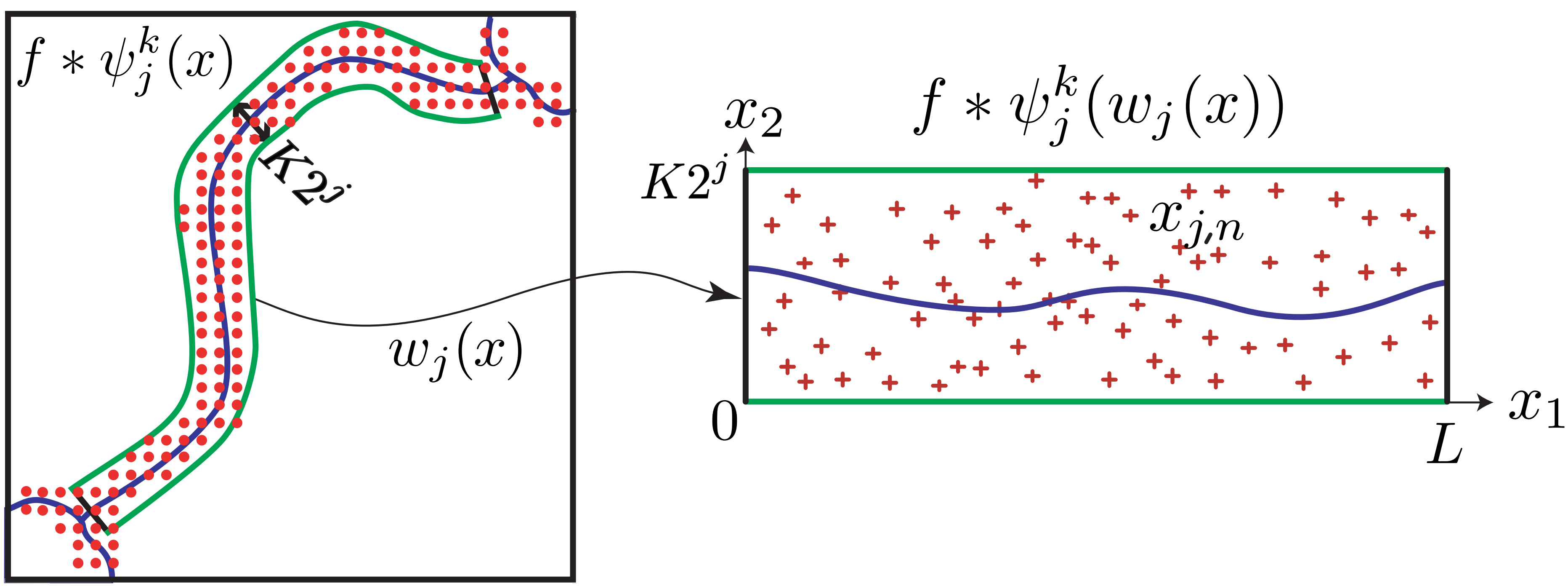


Warping the Wavelet Space

Gabriel Peyré

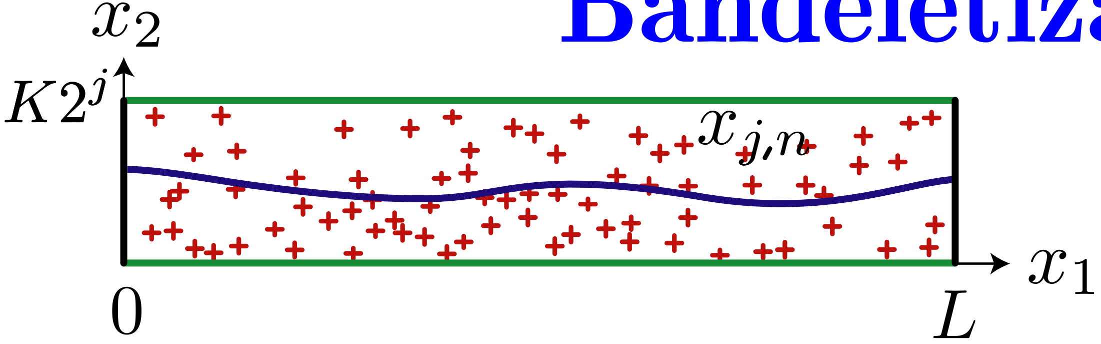
- Wavelet coefficients are samples of a regularized function:

$$\langle f\,,\,\psi_{j,n}^k\rangle = f\star\psi_j^k(2^j n) \quad \text{with} \quad \psi_j^k(x) = 2^{-j}\,\psi^k(-2^{-j}x) \quad .$$



Bandeletization

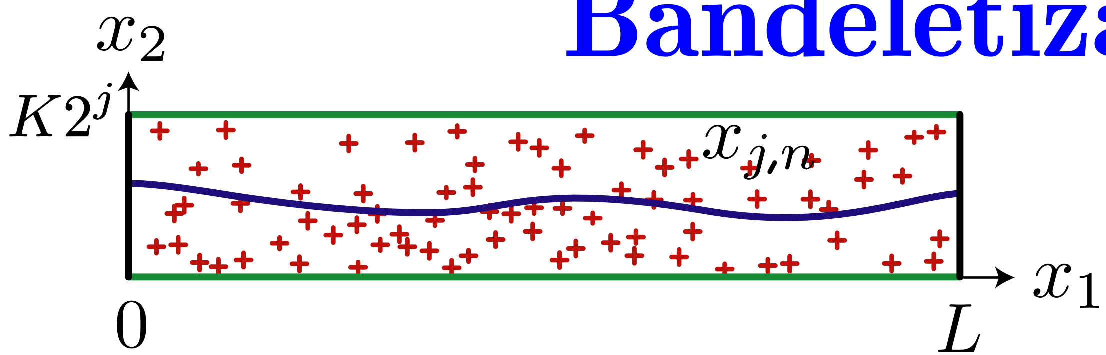
Bandeletization



$$f_j(x) = f \star \psi_j^k(w_j(x))$$

- $$\left| \frac{\partial^{a+b} f_j(x_1, x_2)}{\partial^a x_1 \, \partial^b x_2} \right| \leqslant C \, 2^{-bj} \, 2^{-aj/\alpha} \quad .$$

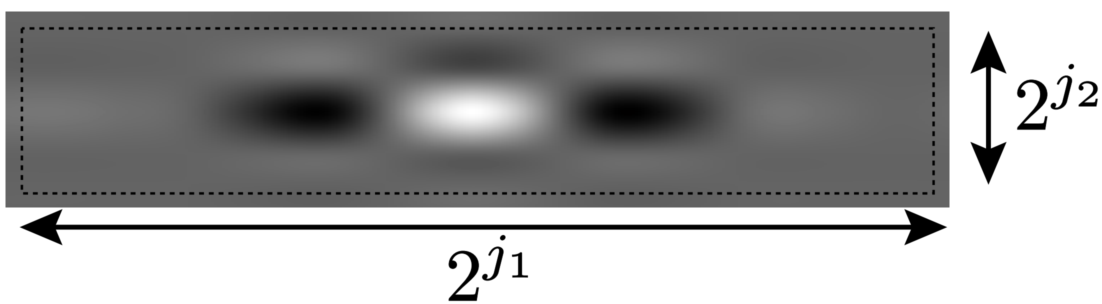
Bandeletization



$$f_j(x) = f \star \psi_j^k(w_j(x))$$

- $$\left| \frac{\partial^{a+b} f_j(x_1, x_2)}{\partial^a x_1 \, \partial^b x_2} \right| \leqslant C \, 2^{-bj} \, 2^{-aj/\alpha} \ .$$

- Approximation from M wavelets of an anisotropic wavelet basis $\{\psi_{j_1,n_1}(x_1)\,\psi_{j_2,n_2}(x_2)\}_{j_1,n_1,j_2,n_2}$:

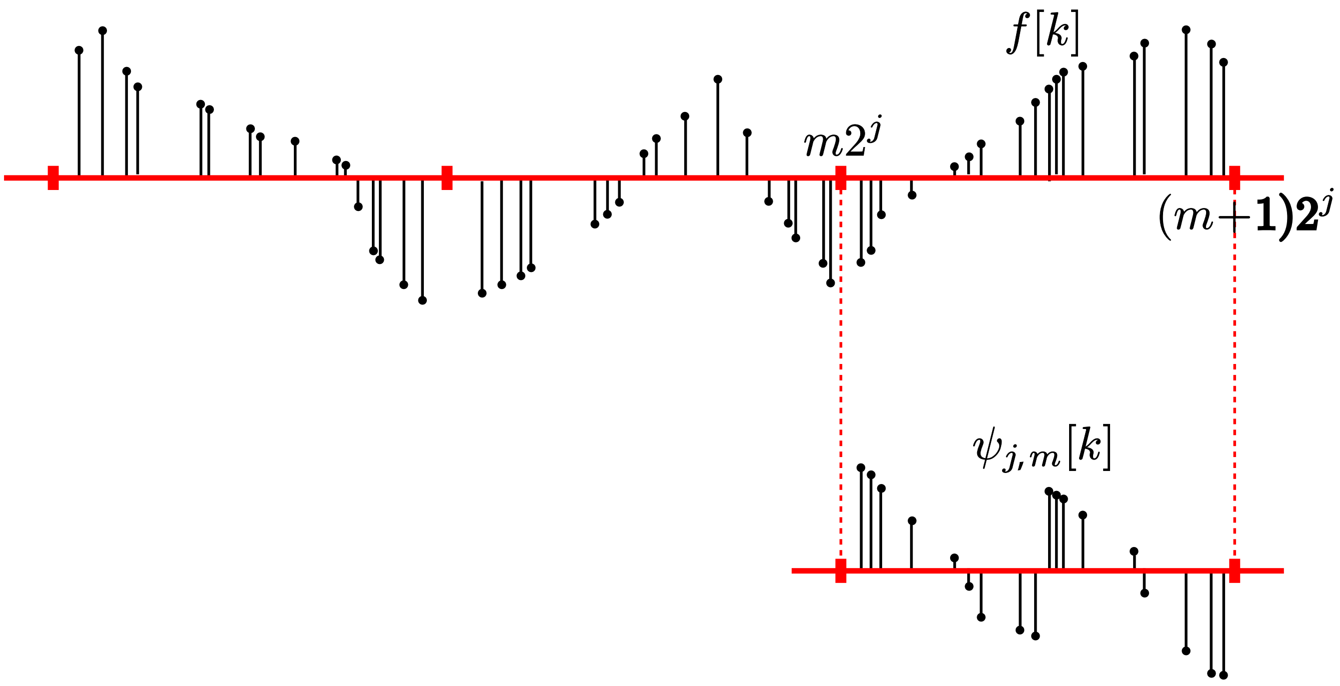


$$\|f_j - f_{j,M}\|^2 \leqslant C\, M^{-\alpha} \ .$$

Irregularly Sampled Alpert Multiwavelets

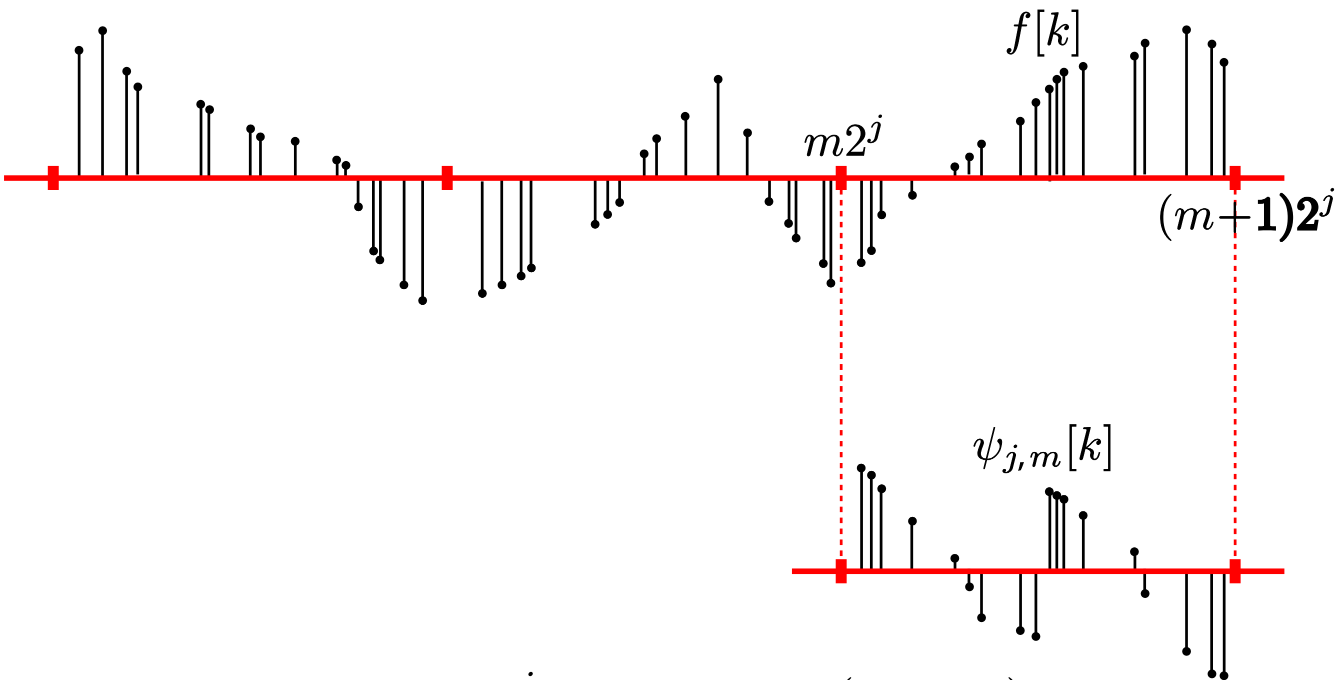
Irregularly Sampled Alpert Multiwavelets

- Alpert discontinuous polynomial multiresolution approximation:
$\mathbf{V}_j = \{f \ : \ f \text{ is a polynomial of degree p on } [m2^j, (m+1)2^j)\}$



Irregularly Sampled Alpert Multiwavelets

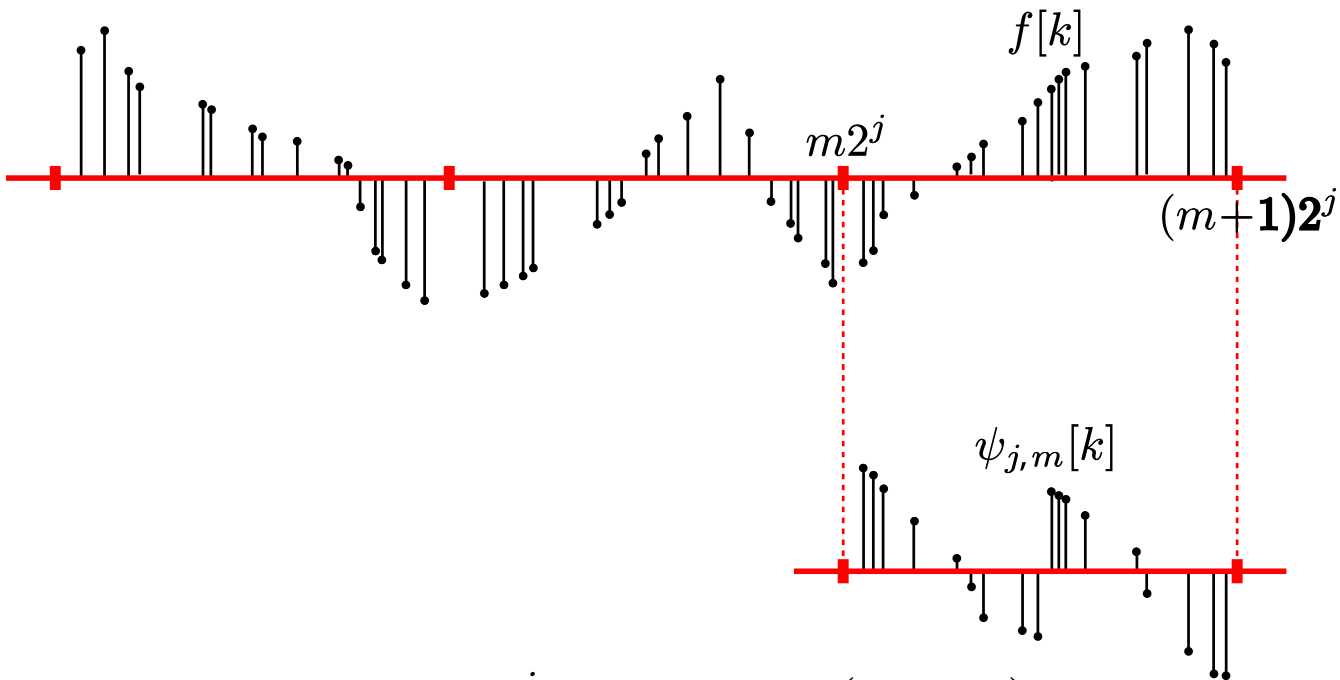
- Alpert discontinuous polynomial multiresolution approximation:
$\mathbf{V}_j = \{f \ : \ f \text{ is a polynomial of degree p on } [m2^j, (m+1)2^j)\}$



- On each interval of size 2^j there are $(p+1)$ wavelets having $(p+1)$ vanishing moments.

Irregularly Sampled Alpert Multiwavelets

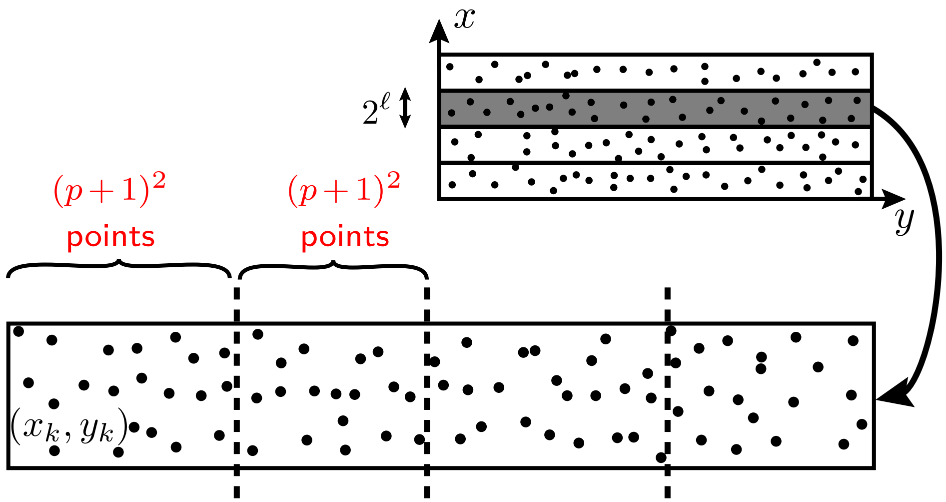
- Alpert discontinuous polynomial multiresolution approximation:
  $\mathbf{V}_j = \{f \ : \ f \text{ is a polynomial of degree p on } [m2^j, (m+1)2^j)\}$



- On each interval of size 2^j there are $(p+1)$ wavelets having $(p+1)$ vanishing moments.
- Alpert fast wavelet transform is $O(N)$ for N irregularly spaced samples.

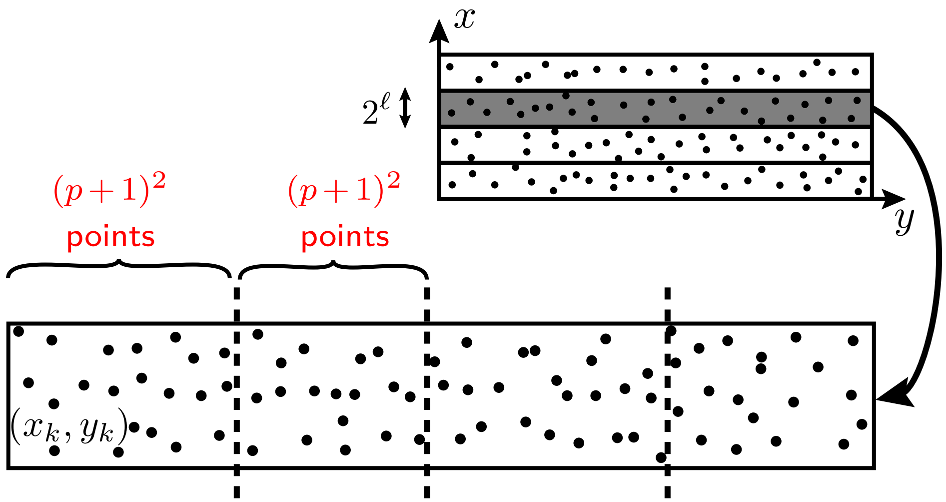
2D Discrete Alpert Multiwavelets

2D Discrete Alpert Multiwavelets



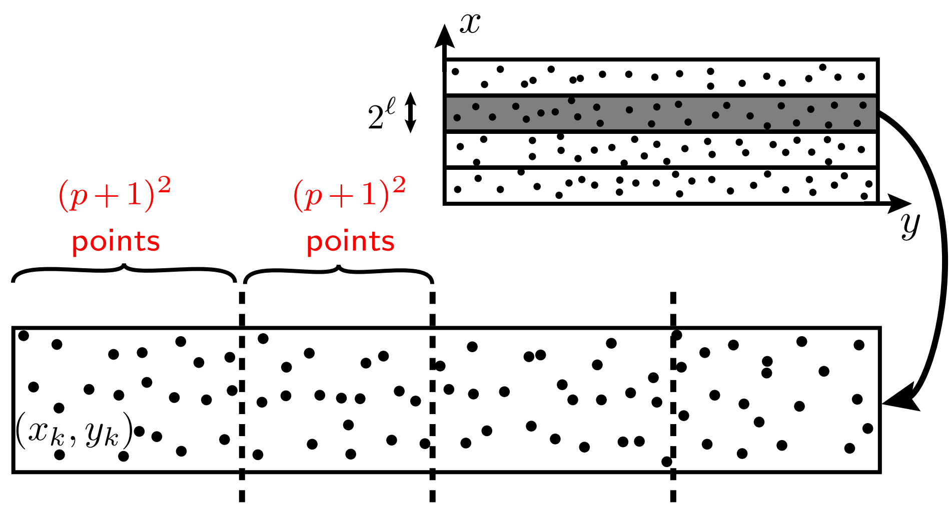
- On each slice take basis vectors $(x_k^i y_k^j)$ for $i, j = 0 \ldots p$.

2D Discrete Alpert Multiwavelets



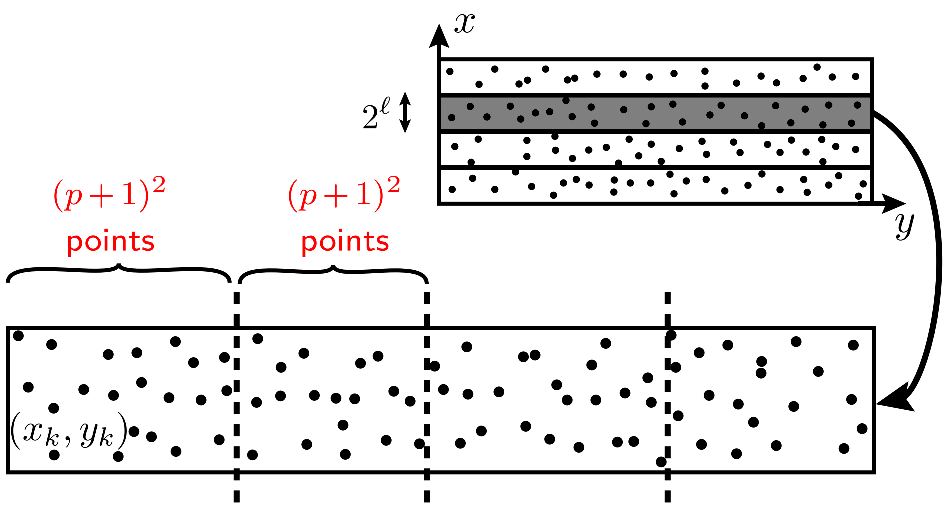
- On each slice take basis vectors $(x_k^i y_k^j)$ for $i, j = 0 \ldots p$.
- On each slice same 1D fast $O(n)$ algorithm.

2D Discrete Alpert Multiwavelets

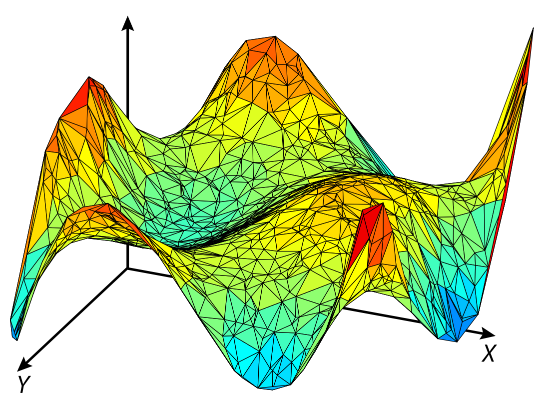


- On each slice take basis vectors $(x_k^i y_k^j)$ for $i, j = 0 \dots p$.
- On each slice same 1D fast $O(n)$ algorithm.
- Stable with respect to sampling location.

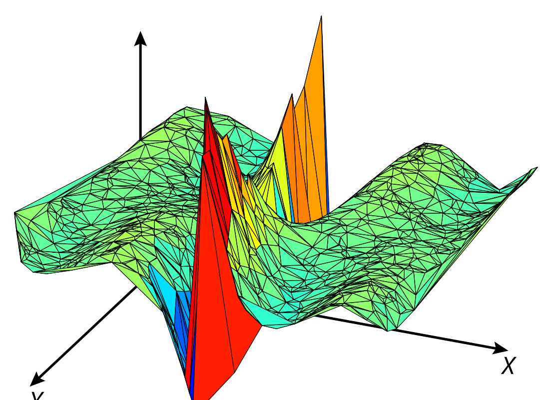
2D Discrete Alpert Multiwavelets



- On each slice take basis vectors $(x_k^i y_k^j)$ for $i, j = 0 \dots p$.
- On each slice same 1D fast $O(n)$ algorithm.
- Stable with respect to sampling location.



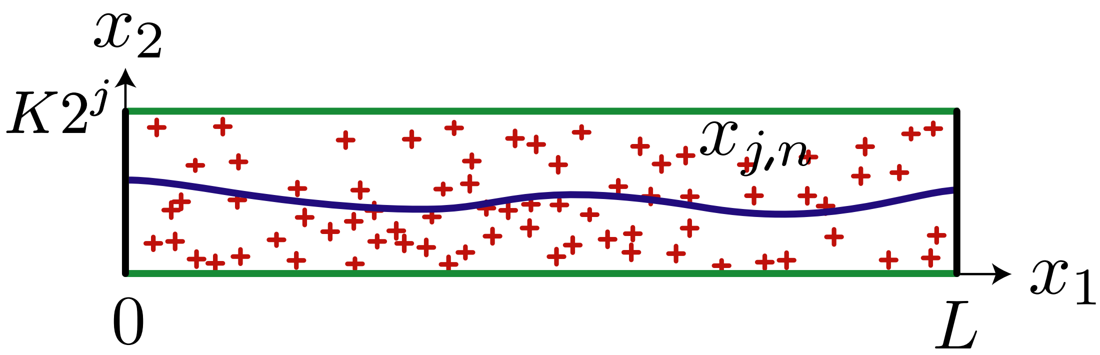
Scaling function



Wavelet function

Bandeletization with 2D Alpert Wavelets

Bandeletization with 2D Alpert Wavelets

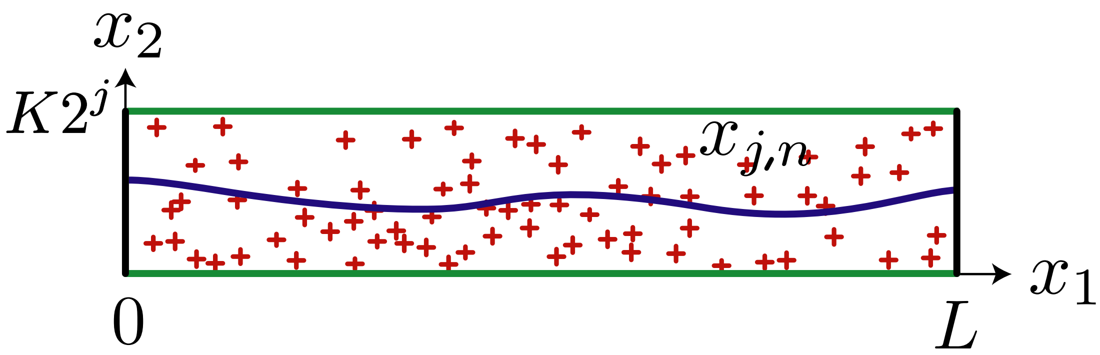


$\bar{f}_j[n] = f_j(x_{j,n}) \in^{N_j}$

- Approximation of $\bar{f}_j[n]$ in a 2D anisotropic Alpert wavelet basis $\{a_{j,m}[n]\}_{0 \leqslant n < N_j}$:

$$\bar{f}_{j,M}[n] = \sum_{|\langle \bar{f}_j, a_m \rangle| > T_M} \langle \bar{f}_j, a_{j,m} \rangle \, a_{j,m}[n] \ .$$

Bandeletization with 2D Alpert Wavelets



$\bar{f}_j[n] = f_j(x_{j,n}) \in^{N_j}$

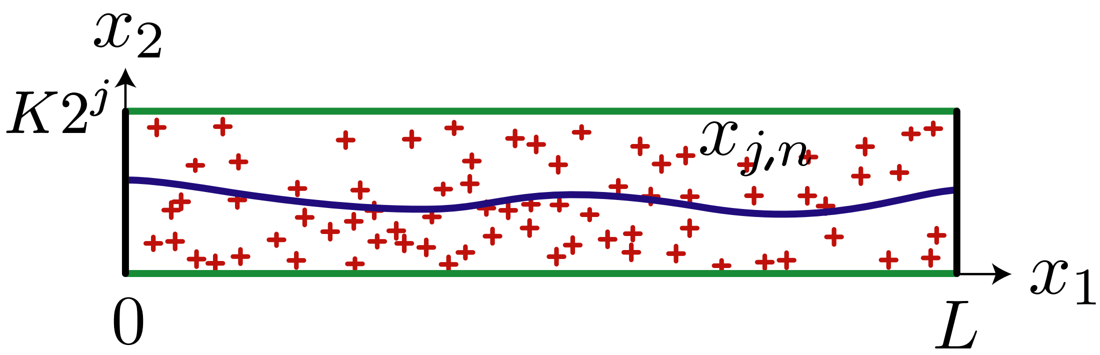
- Approximation of $\bar{f}_j[n]$ in a 2D anisotropic Alpert wavelet basis $\{a_{j,m}[n]\}_{0 \leqslant n < N_j}$:

$$\bar{f}_{j,M}[n] = \sum_{|\langle \bar{f}_j, a_m \rangle| > T_M} \langle \bar{f}_j, a_{j,m} \rangle \, a_{j,m}[n] \ .$$

- Requires $O(N_j)$ operations and

$$\|\bar{f}_j - \bar{f}_{j,M}\|^2 \leqslant C \, M^{-\alpha} \ .$$

Bandeletization with 2D Alpert Wavelets



$$\bar{f}_j[n] = f_j(x_{j,n}) \in^{N_j}$$

- Approximation of $\bar{f}_j[n]$ in a 2D anisotropic Alpert wavelet basis $\{a_{j,m}[n]\}_{0 \leqslant n < N_j}$:

$$\bar{f}_{j,M}[n] = \sum_{|\langle \bar{f}_j, a_m \rangle| > T_M} \langle \bar{f}_j, a_{j,m} \rangle \, a_{j,m}[n] \ .$$

- Requires $O(N_j)$ operations and

$$\|\bar{f}_j - \bar{f}_{j,M}\|^2 \leqslant C \, M^{-\alpha} \ .$$

- Similar to V2 neurons.

Second Generation Bandelets

Second Generation Bandelets

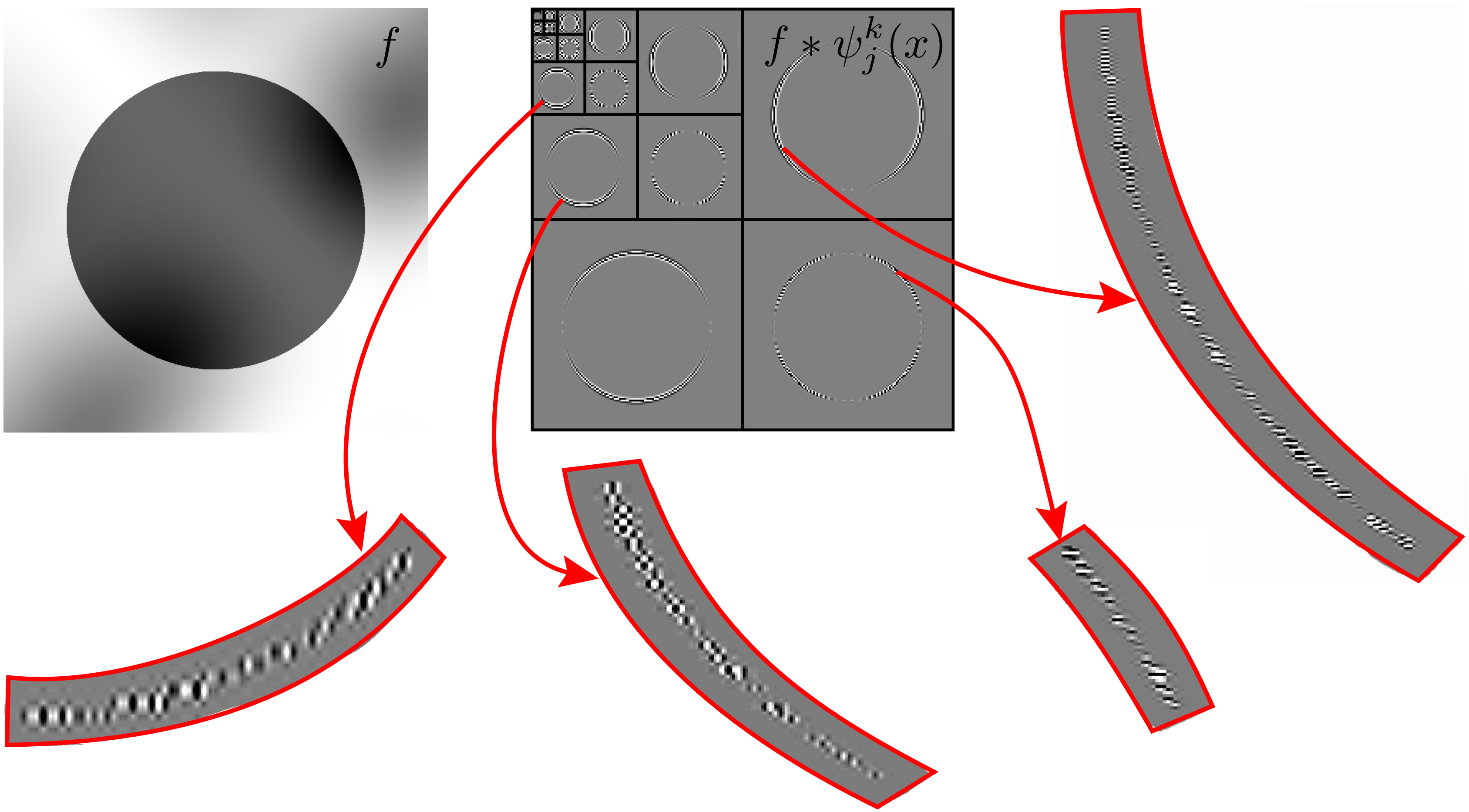
- $$b^k_{j,m}(x) = \sum_{n=1}^{N_j} a_{j,m}[n]\, \psi^k_{j,n}(x)\ .$$

Second Generation Bandelets

- $$b^k_{j,m}(x) = \sum_{n=1}^{N_j} a_{j,m}[n]\, \psi^k_{j,n}(x) \ .$$

Second Generation Bandelets

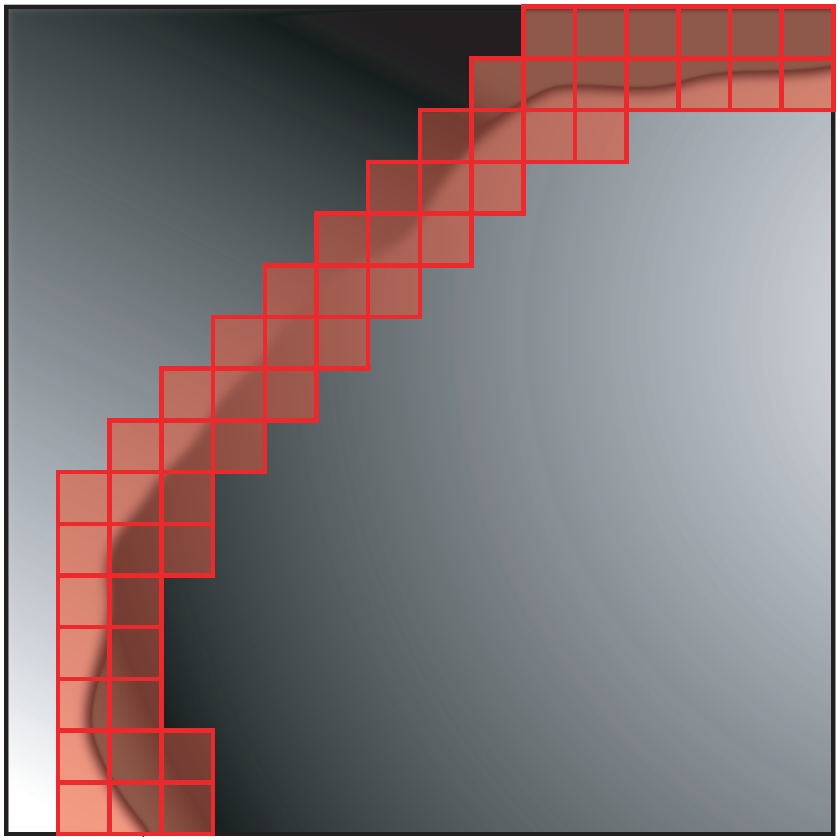
- $$b_{j,m}^k(x) = \sum_{n=1}^{N_j} a_{j,m}[n]\, \psi_{j,n}^k(x)\ .$$
- Bandelet orthonormal basis: $\left\{\psi_{j,n}^k\right\}_{k,j,n} \cup \left\{b_{j,m}^k\right\}_{k,j,m}$.

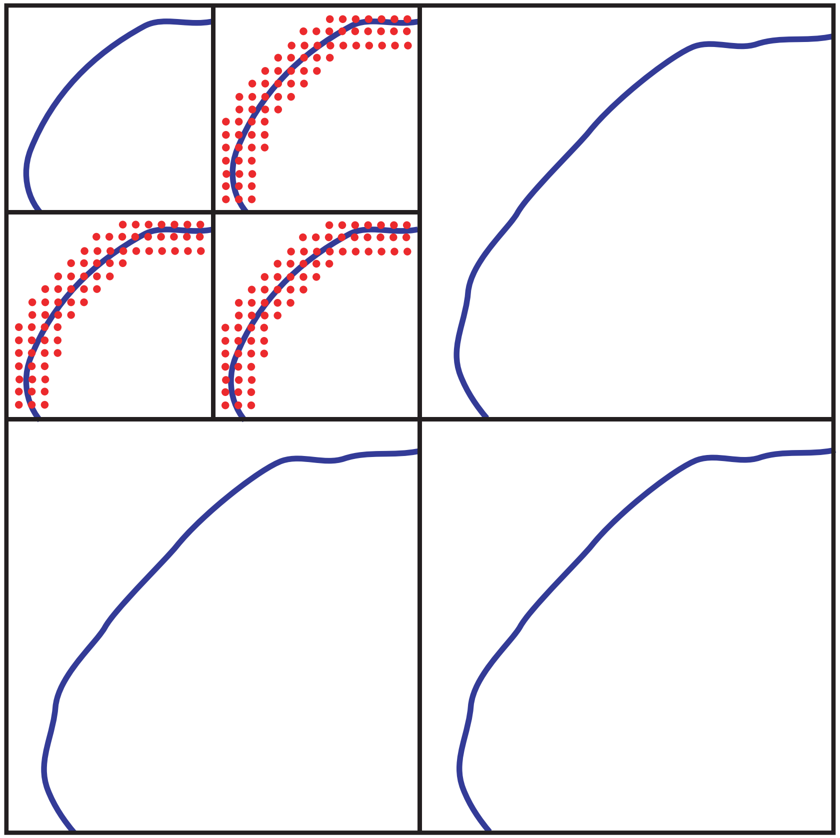


Multiscale Geometry

Multiscale Geometry

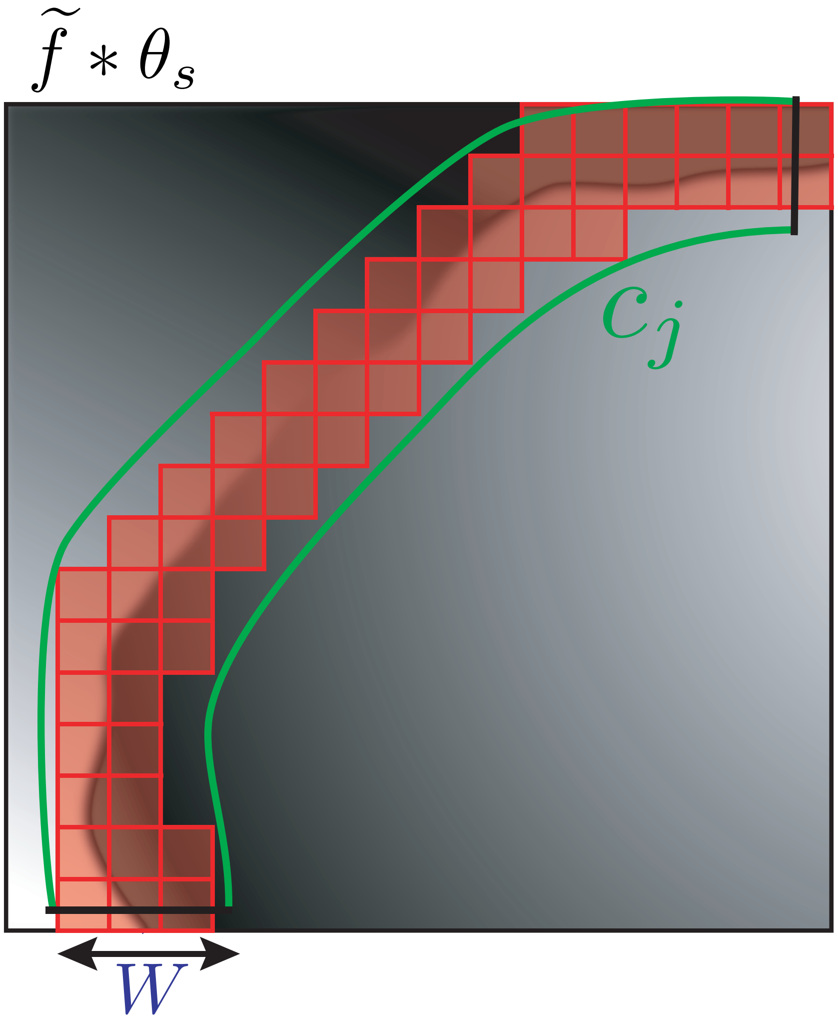
$\widetilde{f} * \theta_s$

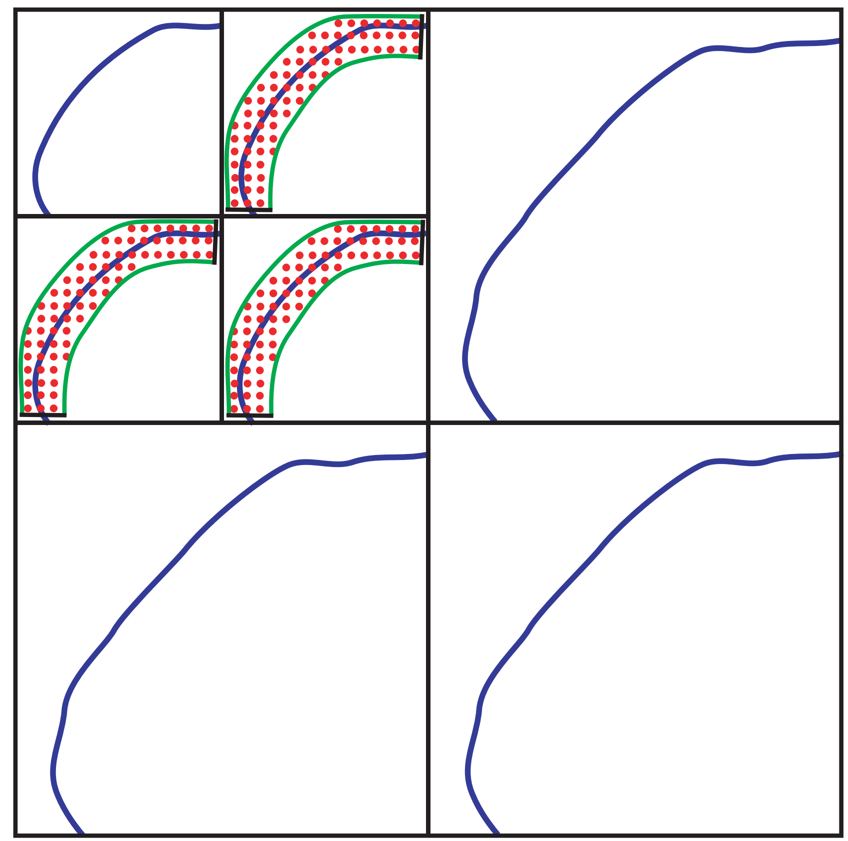




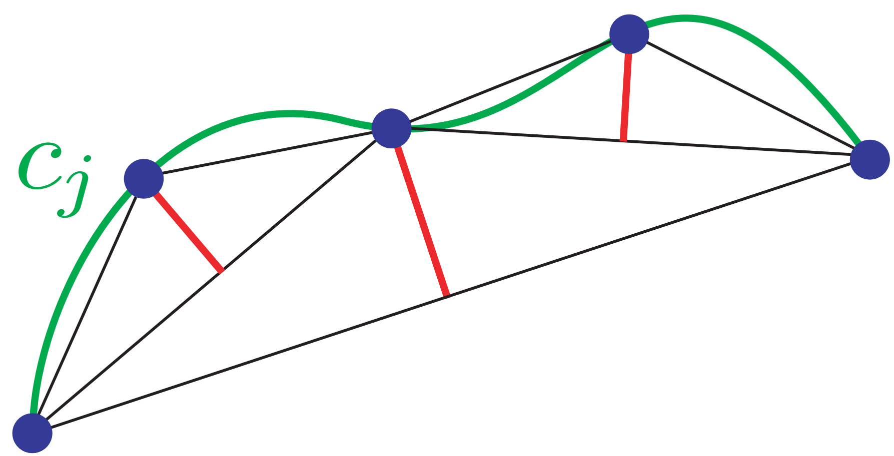
- detection threshold D_j

Multiscale Geometry



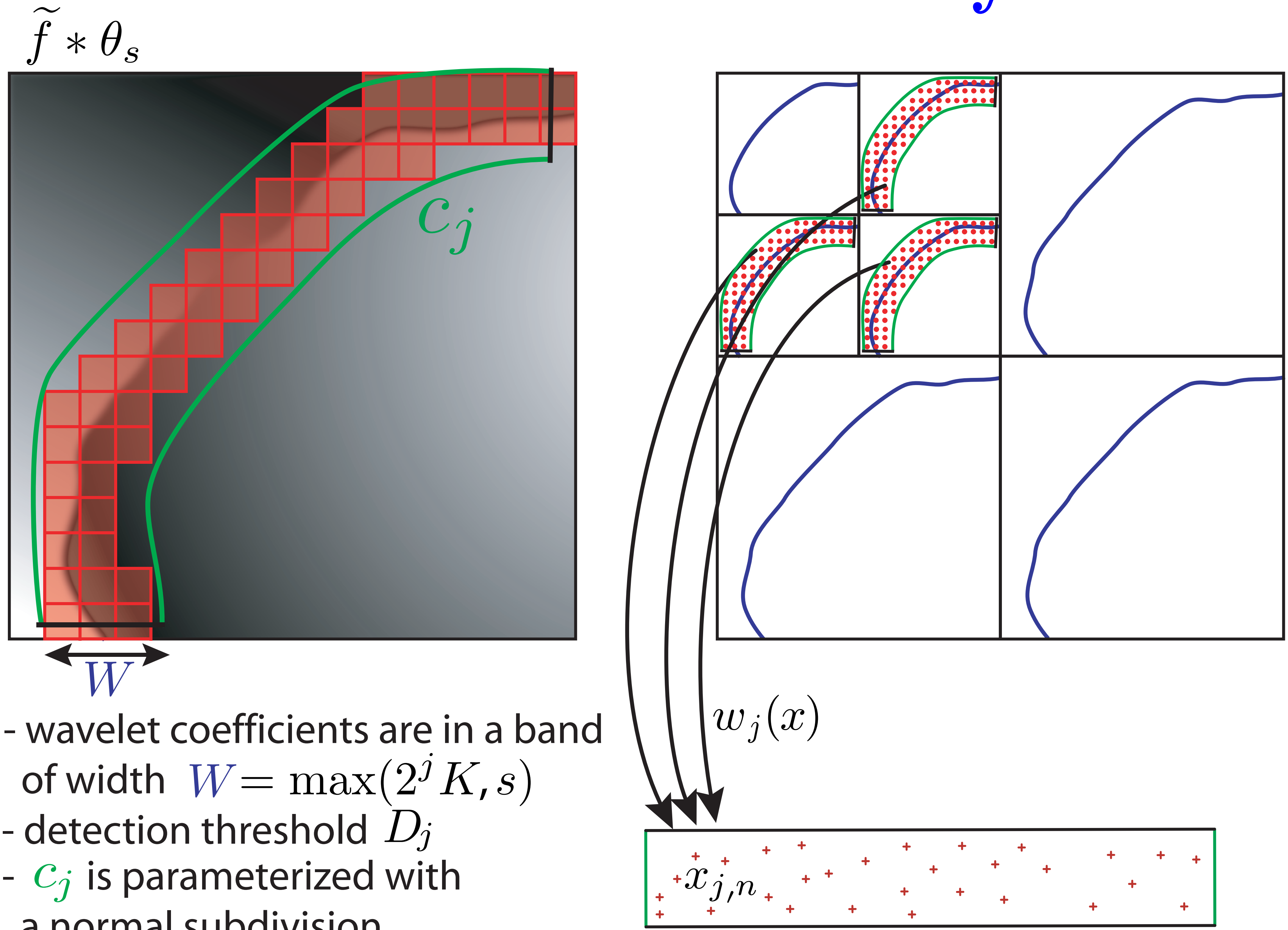


- wavelet coefficients are in a band of width $W = \max(2^j K, s)$
- detection threshold D_j
- c_j is parameterized with a normal subdivision



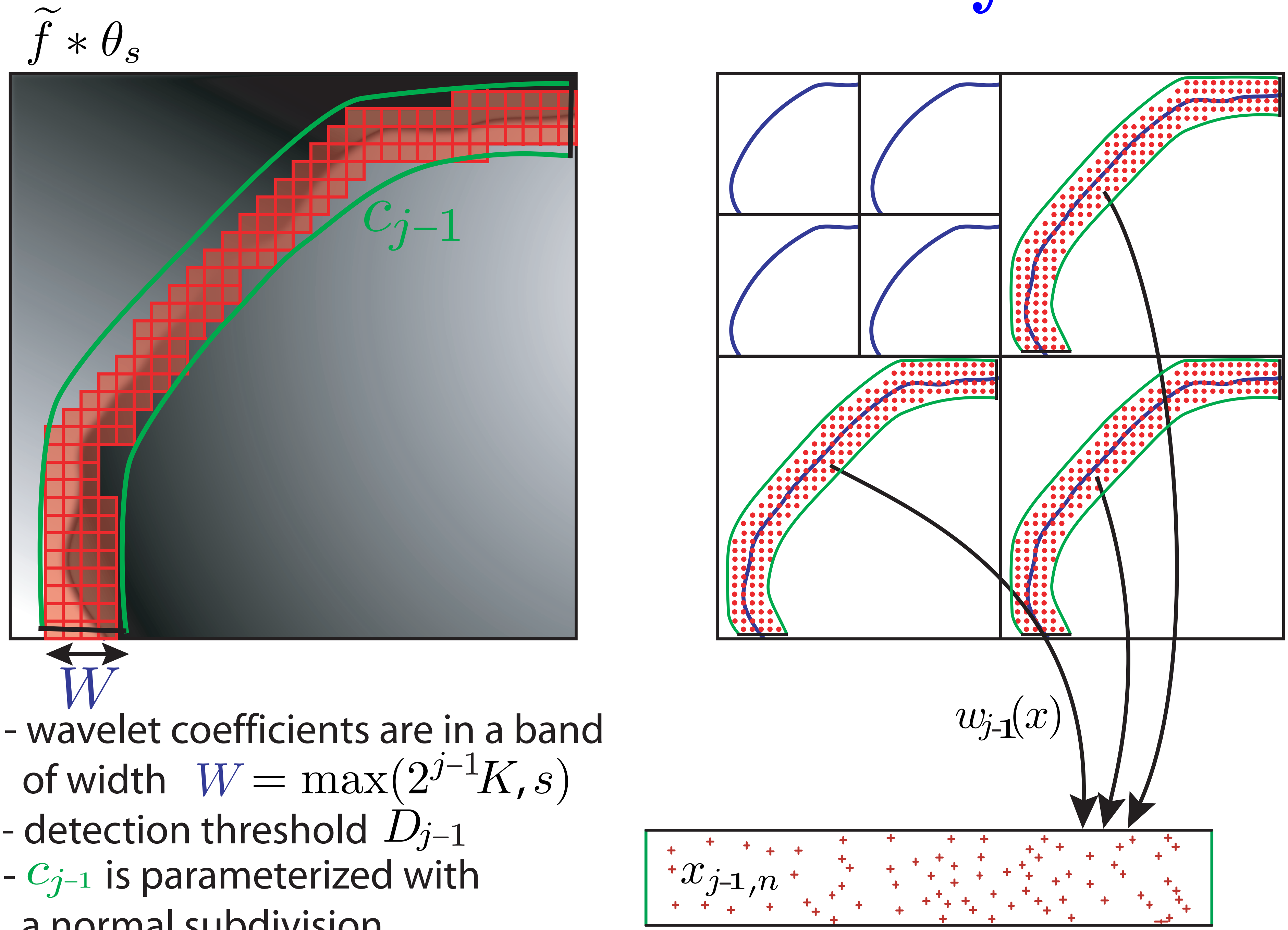
[*Daubechies, Runborg, Sweldens*]

Multiscale Geometry



- wavelet coefficients are in a band of width $W = \max(2^j K, s)$
- detection threshold D_j
- c_j is parameterized with a normal subdivision [Daubechies, Runborg, Sweldens]

Multiscale Geometry

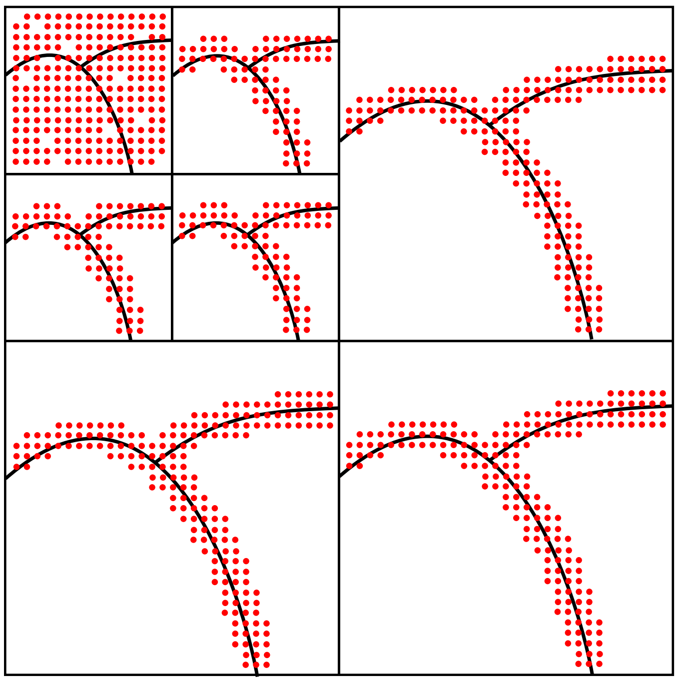


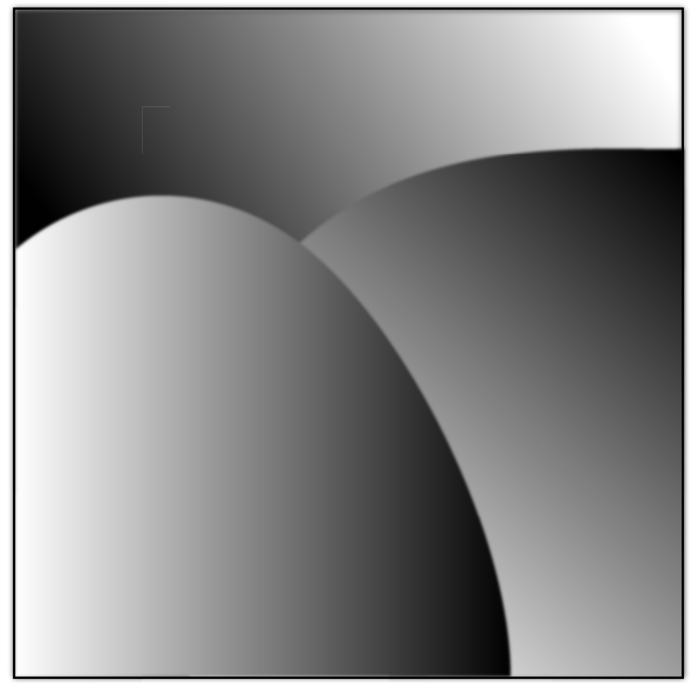
- wavelet coefficients are in a band of width $W = \max(2^{j-1}K, s)$
- detection threshold D_{j-1}
- c_{j-1} is parameterized with a normal subdivision

[*Daubechies, Runborg, Sweldens*]

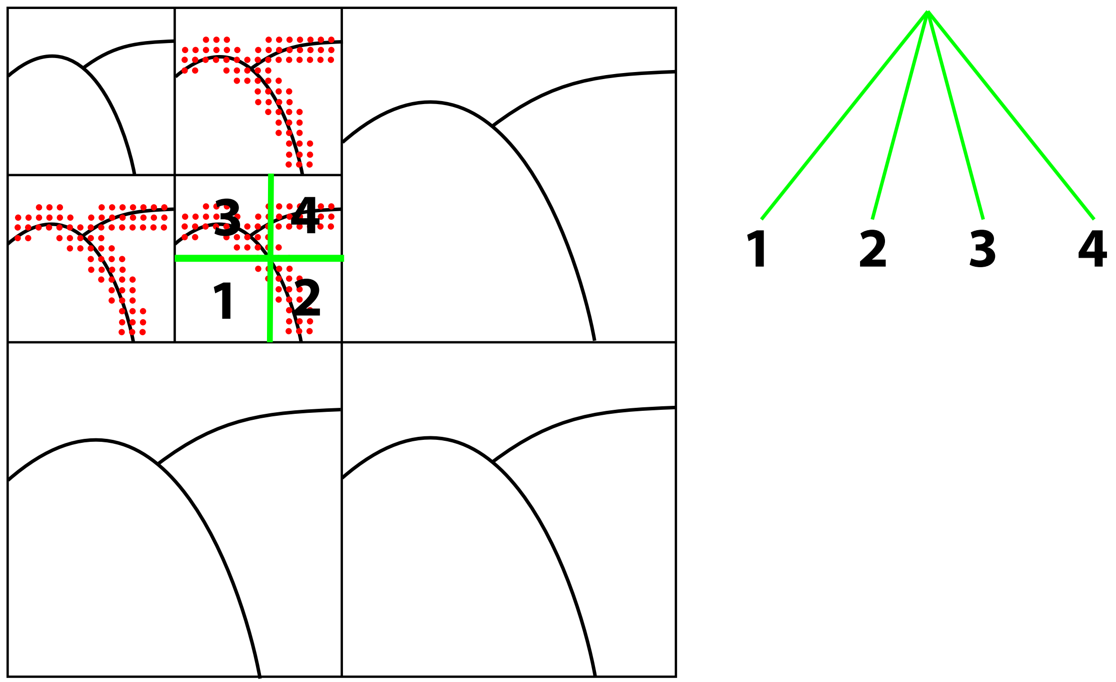
Dyadic Segmentation

Dyadic Segmentation

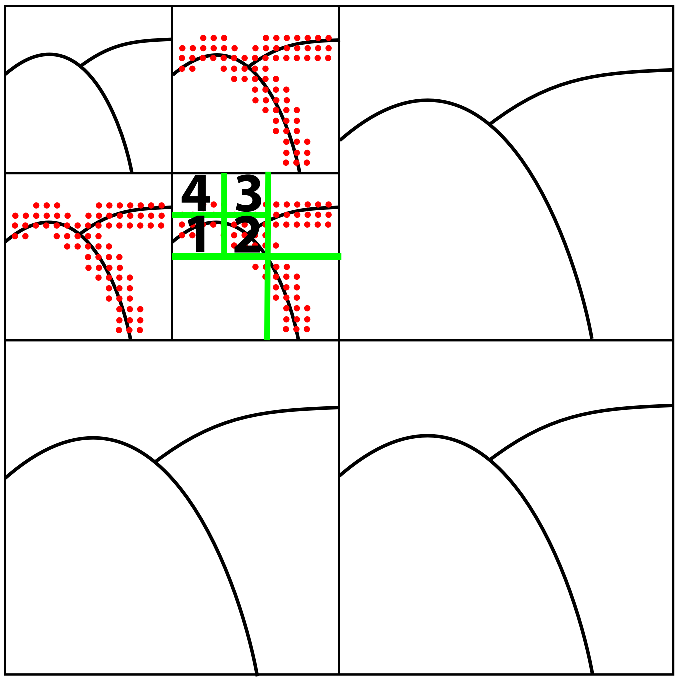


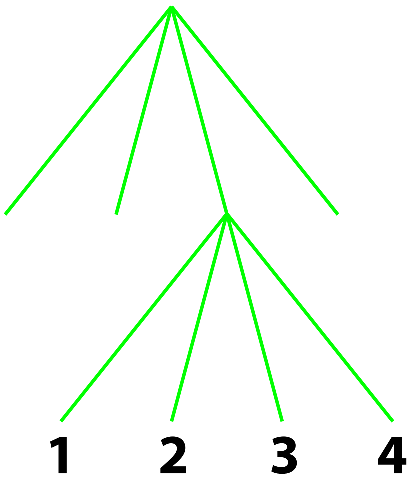


Dyadic Segmentation

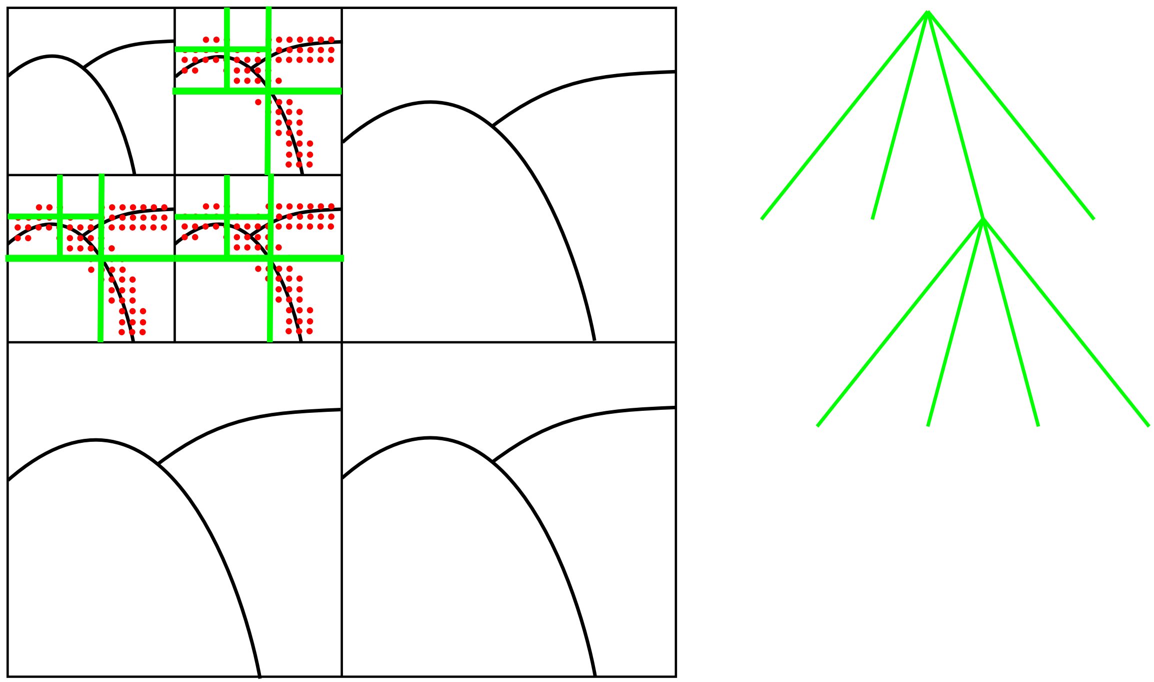


Dyadic Segmentation

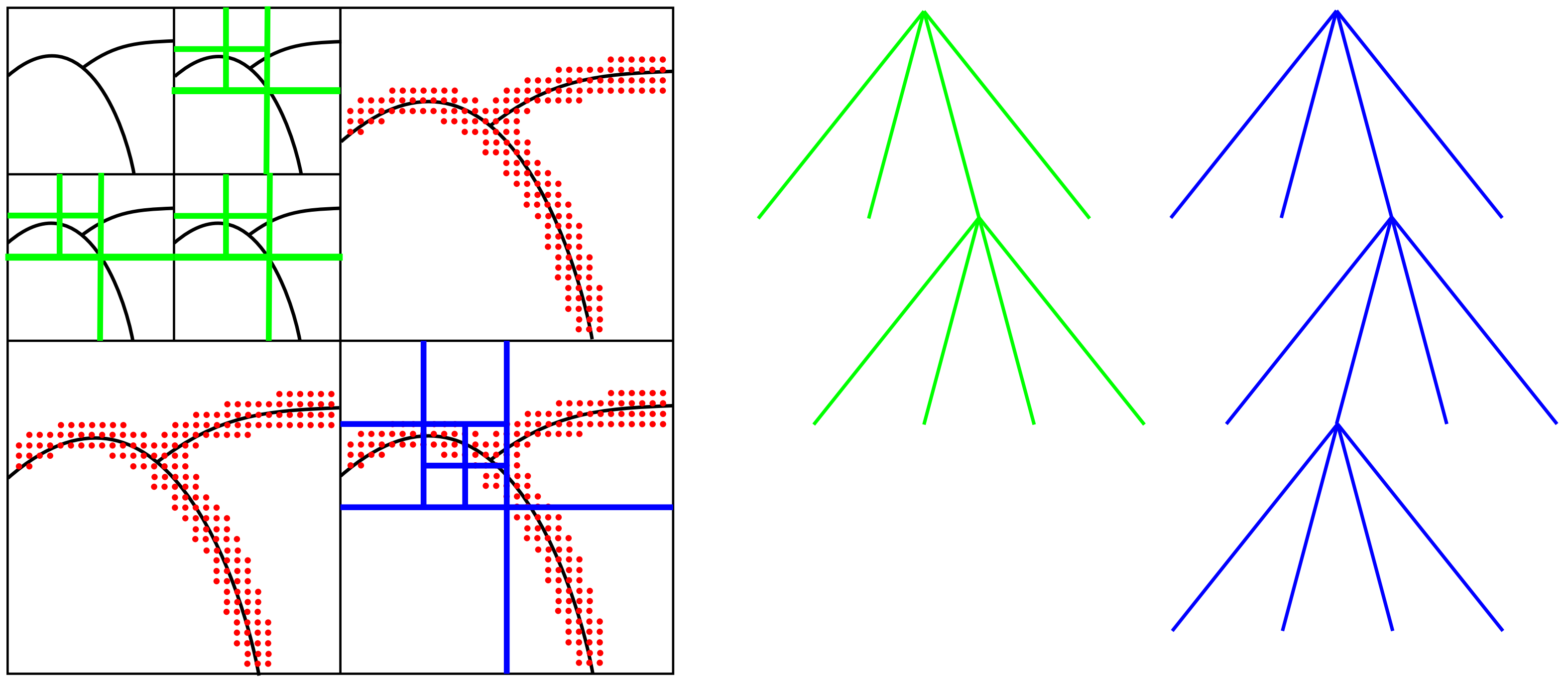




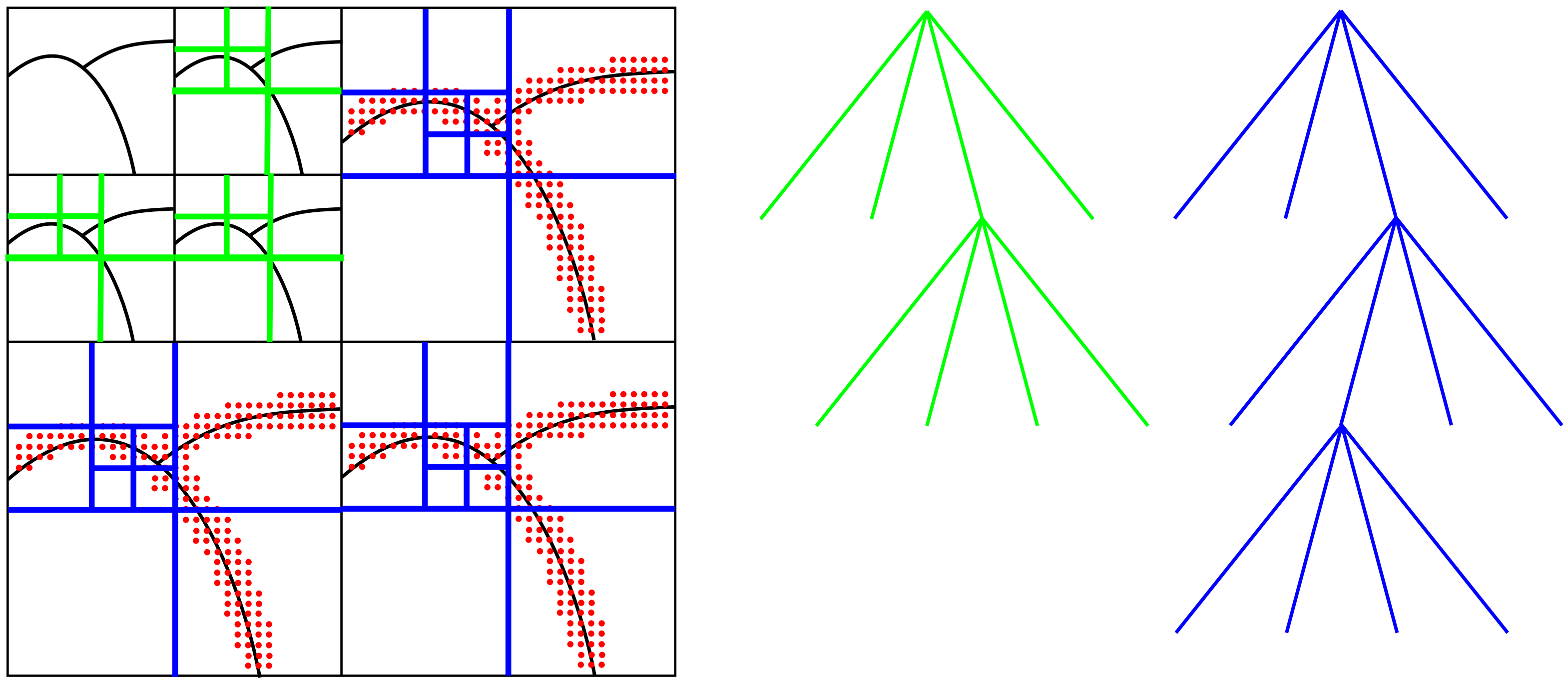
Dyadic Segmentation



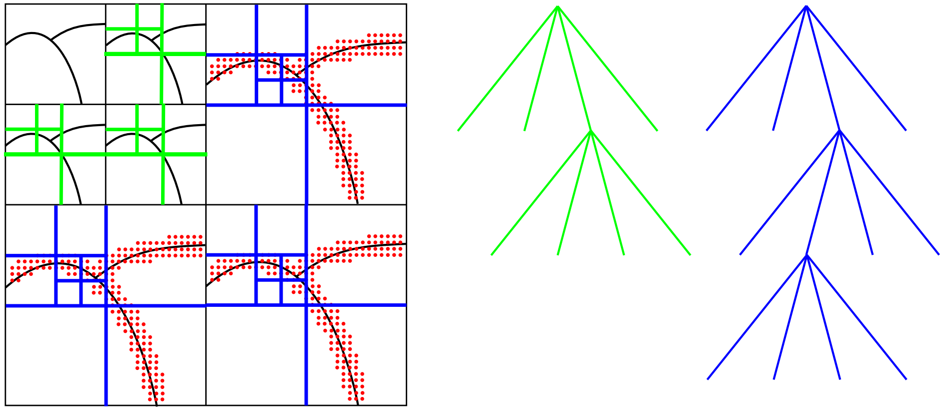
Dyadic Segmentation



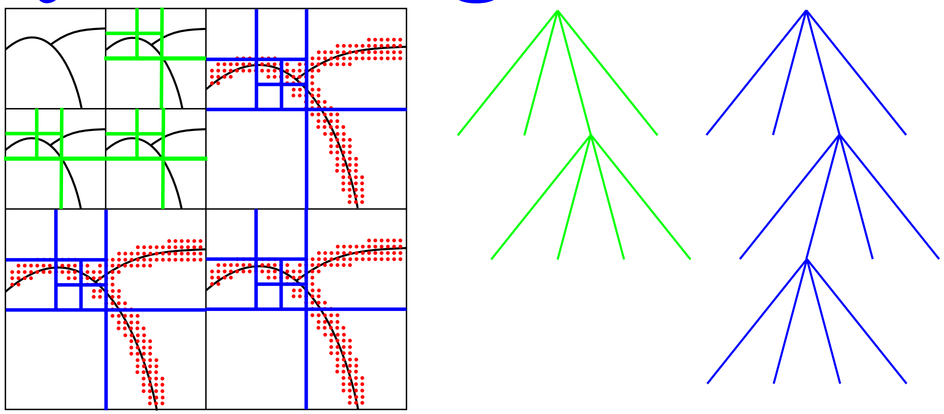
Dyadic Segmentation



Dyadic Segmentation



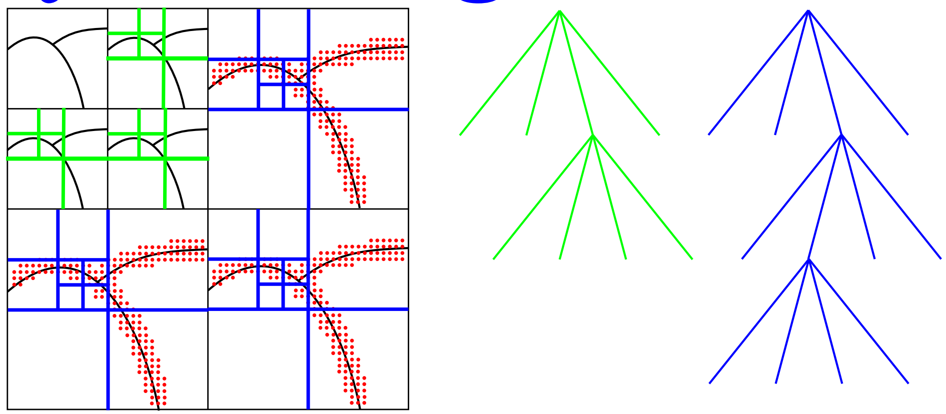
Dyadic Segmentation



- Total number of bandelet, wavelet and geometric coefficients:

$$M = \sum_j M_j = \sum_j \Big(M_{B_j} + M_{W_j} + M_{G_j} \Big)$$

Dyadic Segmentation



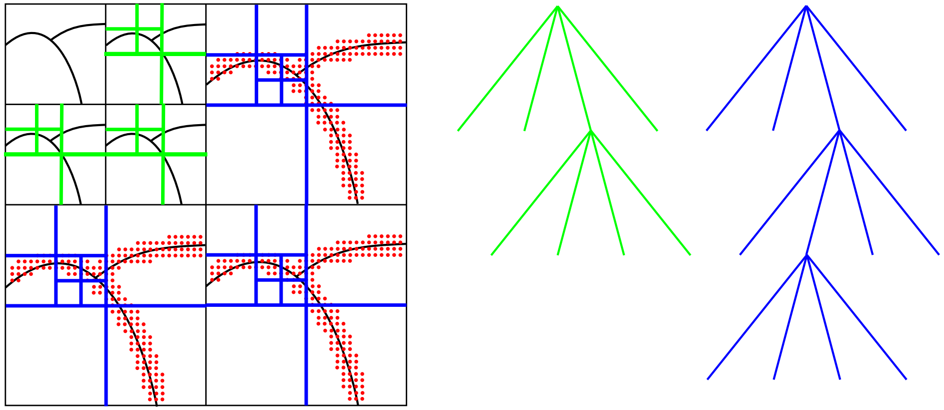
- Total number of bandelet, wavelet and geometric coefficients:

$$M = \sum_j M_j = \sum_j \Big(M_{B_j} + M_{W_j} + M_{G_j} \Big)$$

- Best basis (geometry) computed by minimizing $\|f - f_M\|^2 + T^2 M$:

$$\|f - f_M\|^2 + T^2 M = \sum_j \left(\sum_{|b_{j,m}|<T} |b_{j,m}|^2 + \sum_{|w_{j,m}|<T} |w_{j,m}|^2 + T^2 M_j \right)$$

Dyadic Segmentation



- Total number of bandelet, wavelet and geometric coefficients:

$$M = \sum_j M_j = \sum_j \left(M_{B_j} + M_{W_j} + M_{G_j} \right)$$

- Best basis (geometry) computed by minimizing $\|f - f_M\|^2 + T^2 M$:

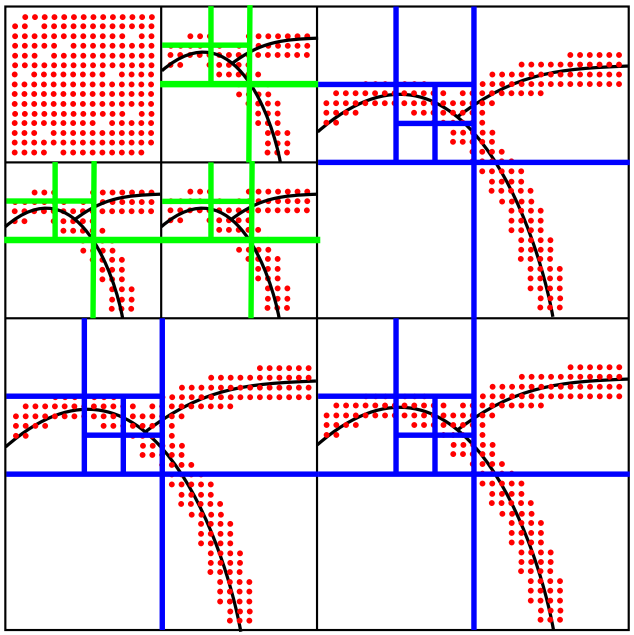
$$\|f - f_M\|^2 + T^2 M = \sum_j \left(\sum_{|b_{j,m}|<T} |b_{j,m}|^2 + \sum_{|w_{j,m}|<T} |w_{j,m}|^2 + T^2 M_j \right)$$

- Computed with $O(N \log_2 N)$ operations with a CART algorithm

Bandelet Representation

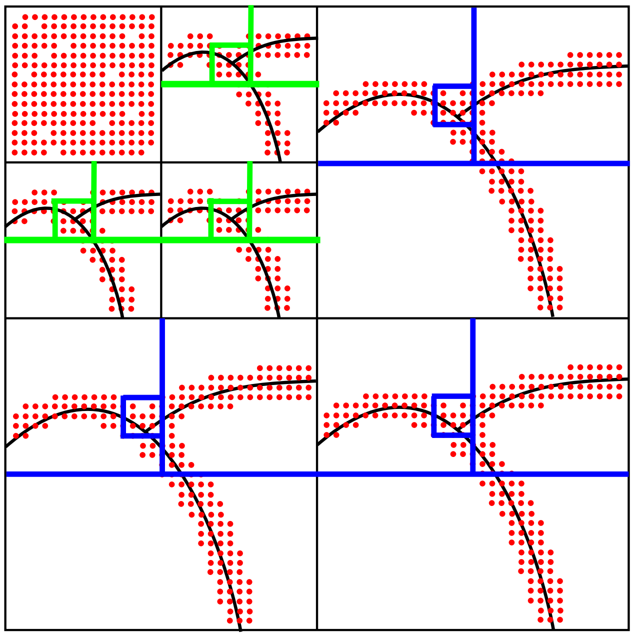
Bandelet Representation

- Neighbor square regions are unified if it decreases $\|f - f_M\|^2 + T^2 M$.



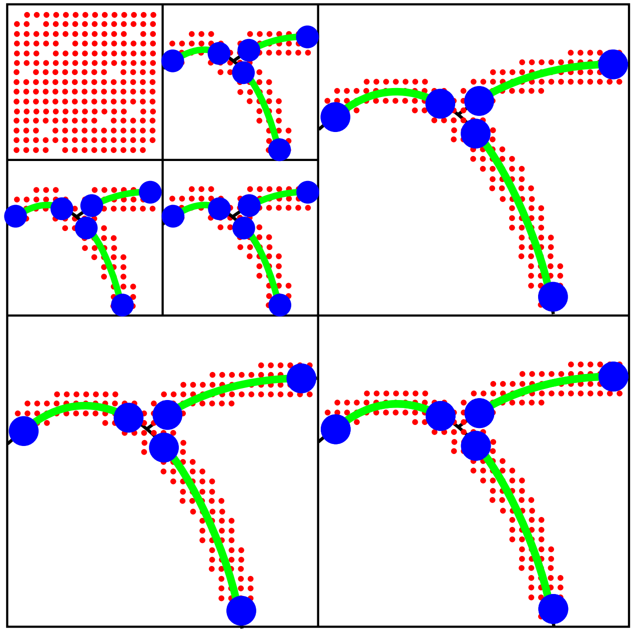
Bandelet Representation

- Neighbor square regions are unified if it decreases $\|f - f_M\|^2 + T^2 M$.



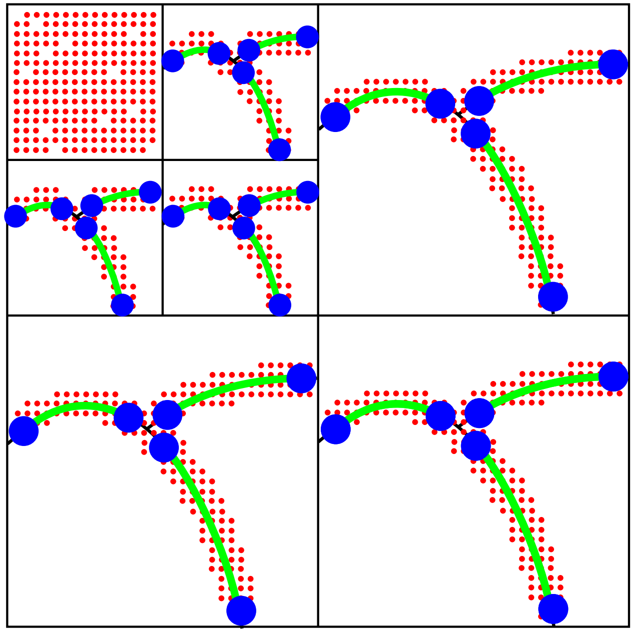
Bandelet Representation

- Neighbor square regions are unified if it decreases $\|f - f_M\|^2 + T^2 M$.



Bandelet Representation

- Neighbor square regions are unified if it decreases $\|f - f_M\|^2 + T^2 M$.



- A bandelet representation includes:
  - Beginning and ending points of bands at each scale.
  - Geometric wavelet coefficients that specify each band.
  - Bandelet coefficients in each band.
  - Wavelet coefficients outside all bands.

Bandelet Approximation Theorem

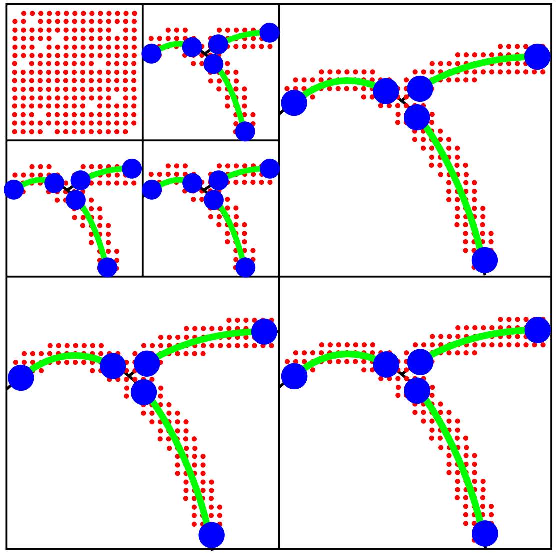
Gabriel Peyré

Bandelet Approximation Theorem

Gabriel Peyré

Theorem: Suppose that $\tilde{f}$ is $\mathbf{C}^\alpha$ away from "edges" that are piecewise $\mathbf{C}^\alpha$.
If $f = \tilde{f}$ or $f = \tilde{f} \star \theta_s$ then a bandelet approximation f_M, with $M = M_B + M_W + M_G$, satisfies

$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$

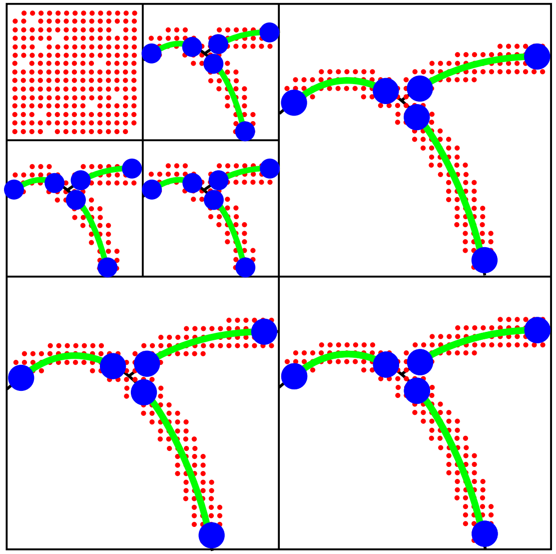


Bandelet Approximation Theorem

Gabriel Peyré

Theorem: Suppose that $\tilde{f}$ is $\mathbf{C}^\alpha$ away from "edges" that are piecewise $\mathbf{C}^\alpha$.
If $f = \tilde{f}$ or $f = \tilde{f} \star \theta_s$ then a bandelet approximation f_M, with $M = M_B + M_W + M_G$, satisfies

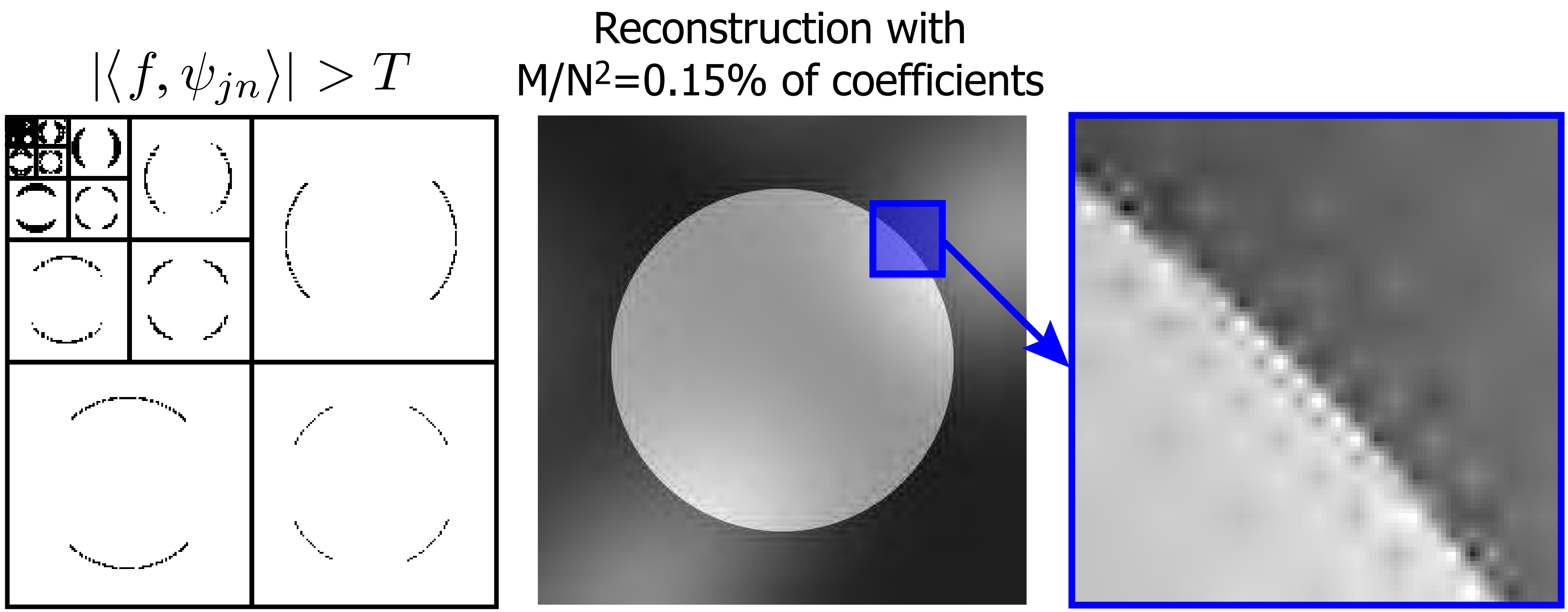
$$\|f - f_M\|^2 \leqslant C\, M^{-\alpha} \ .$$



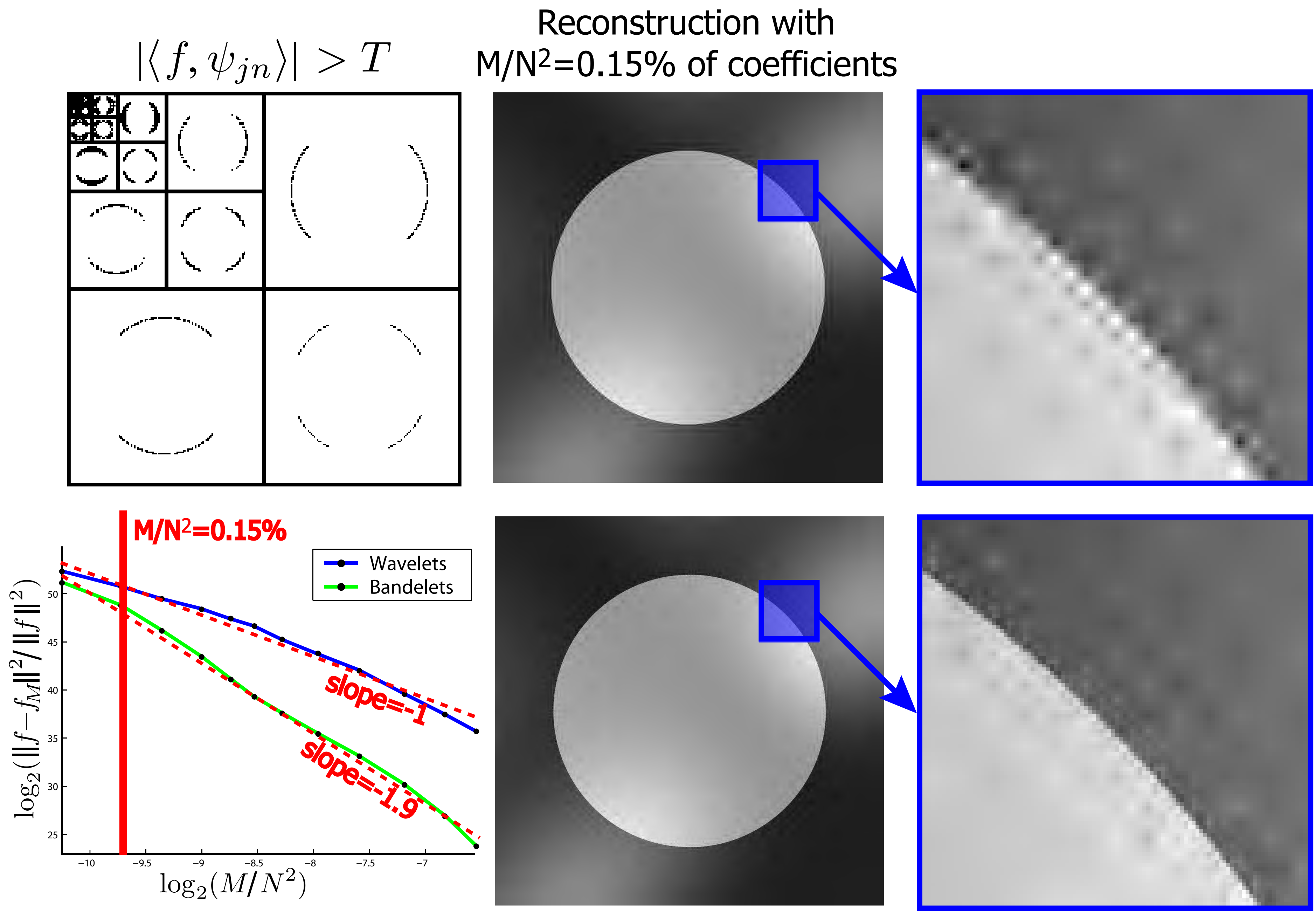
- Optimal (unknown) decay exponent α.

Numerical Experiments

Numerical Experiments



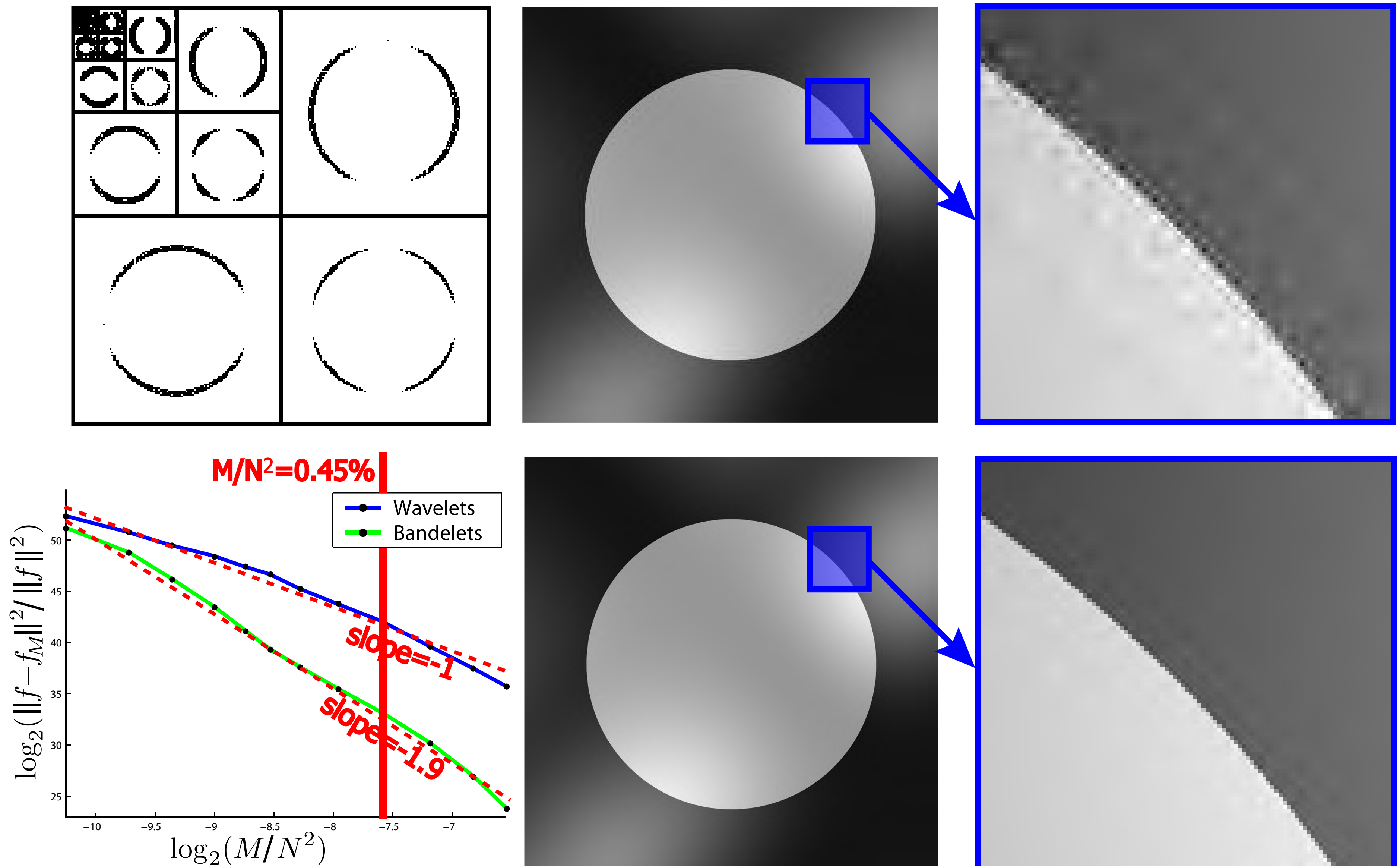
Numerical Experiments



Numerical Experiments

$|\langle f, \psi_{jn} \rangle| > T$

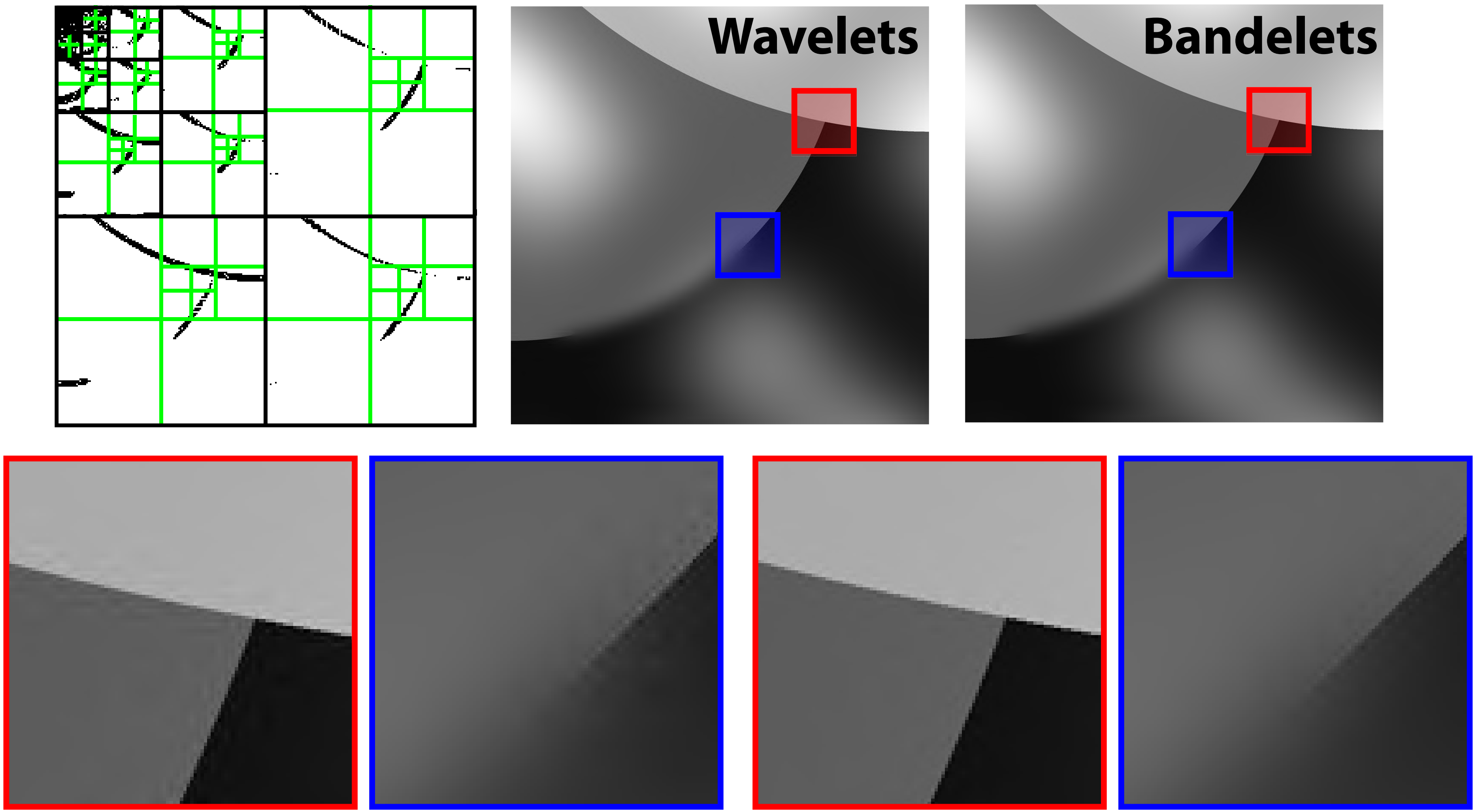
Reconstruction with M/N^2=0.45% of coefficients



Numerical Experiments

$|\langle f, \psi_{jn} \rangle| > T$

Reconstruction with M/N^2=0.45% of coefficients



Wavelets

Bandelets

Application to ID Photos Compression

Application to ID Photos Compression

- ID Photo: easy way of authentification. 

Application to ID Photos Compression

- ID Photo: easy way of authentification.

- But easy to forge.

Application to ID Photos Compression

- ID Photo: easy way of authentification. 
- But easy to forge.
- Secured solution: digital picture plus cryptology and digital signature.

Application to ID Photos Compression

- ID Photo: easy way of authentification. 
- But easy to forge.
- Secured solution: digital picture plus cryptology and digital signature.
- But limited capacity of storage.

Application to ID Photos Compression

- ID Photo: easy way of authentification.

- But easy to forge.
- Secured solution: digital picture plus cryptology and digital signature.
- But limited capacity of storage.
- 2D Barcode (500-800 bytes), Contactless card (up to 32 kbytes), Smartcard (up to 512 kbytes).

Application to ID Photos Compression

- ID Photo: easy way of authentification. 
- But easy to forge.
- Secured solution: digital picture plus cryptology and digital signature.
- But limited capacity of storage.
- 2D Barcode (500-800 bytes), Contactless card (up to 32 kbytes), Smartcard (up to 512 kbytes).
- LET IT WAVE: image compression codec adapted to the geometry of faces.

ID Photo

ID Photo

500 bytes

ID Photo

500 bytes

JPEG

ID Photo

500 bytes

JPEG

JPEG-2000

ID Photo

500 bytes

JPEG

JPEG-2000

Bandelets
LET IT WAVE

ID Photos

ID Photos

JPEG | JPEG-2000 | LIW

500 bytes

ID Photos

JPEG | JPEG-2000 | LIW

800 bytes

ID Photos

JPEG | JPEG-2000 | LIW

1000 bytes

ID Photos

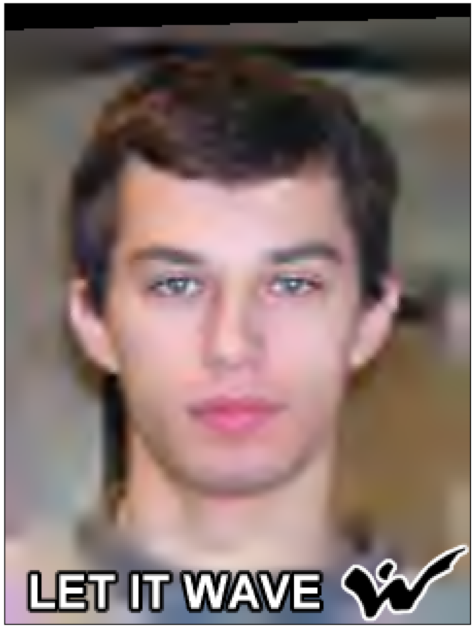
ID Photos

ID Photos

ID Photos

- Complete system: from the camera to the compressed image through a reframing.
- Detection of the face and its geometry.
- Reframing.

ID Photos



- Complete system: from the camera to the compressed image through a reframing.
- Detection of the face and its geometry.
- Reframing.
- Compression (750 bytes).

Conclusion

Conclusion

- Bandelet(s) provides a practical and theoretical framework for image processing.

Conclusion

- Bandelet(s) provides a practical and theoretical framework for image processing.
- Share the same core idea.

Conclusion

- Bandelet(s) provides a practical and theoretical framework for image processing.
- Share the same core idea.
- Numerical and theoretical results but a lot of unsanswered questions.

Conclusion

- Bandelet(s) provides a practical and theoretical framework for image processing.
- Share the same core idea.
- Numerical and theoretical results but a lot of unsanswered questions.
- Huge field of applications.